Just Intervals and Tone Representation in Contemporary Music

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Abstract

Historically, intervals between musical pitches have been understood through two distinct conceptual models: either as *distances* in an imaginary space or as *ratios* between frequencies or string lengths. Each model has its own biases: the distance model is well-suited to constructing abstract pitch geometries, while the ratio model offers insight into an interval's sonic quality and stability. In recent music theory scholarship, the popularity of the distance model has led to the neglect of interval ratios: in this dissertation, I argue that a return to the ratio model offers a deeper understanding of many works of contemporary music which are difficult to analyze with distance-based tools.

One subset of the ratio intervals is of particular interest: the just intervals, intervals with frequencies related by simple whole number ratios like 3/2 or 5/4, have historically been associated with musical consonance. The psychoacoustical properties which give simple just intervals their consonant effect persist for intervals with more complex ratios; such "extended just intonation" has been explored by composers including Harry Partch and Ben Johnston. In addition to their consonance, just intervals strongly imply a harmonic root, providing cognitive support for various forms of tonal centricity.

In order to apply just intonation theory to a broader variety of music, I propose a theory of harmonic perception based on Hugo Riemann's *Tonvorstellung.* I argue that even intervals which are not perfectly just tend to share harmonic properties with the nearest just interval—for example, we hear a piano's tempered major third as essentially equivalent to a just major third, even though the tempered third is slightly wider. I call this process of matching a heard interval to a nearby just interval “tone representation.” The theory is developed into three preference rules for practical application—these rules draw on recent research in Gestalt and music psychology.

The theory is used to analyze twentieth-century works by composers including Arnold Schoenberg, György Ligeti, La Monte Young, and Gérard Grisey. As a pragmatic theory based in perception, tone representation can illustrate the common factors between works by composers with very different aesthetics and compositional techniques.

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Preface

Since the beginning of Western music theory, two ways of conceptualizing interval have shared an uneasy coexistence—one model conceives of intervals as *ratios* between frequencies or string lengths, emphasizing pure tunings and acoustical relationships, while the other model conceives of intervals as *distances* in space, focusing on relative sizes and abstract geometries. In recent years, theorists have tended to focus on the distance model at the expense of the ratio model—pitch-class set theory, for example, defines interval as a mathematical distance, but is silent on the actual sonic properties of each interval.

In this dissertation, I argue for a revival of the ratio model through the concept of tone representation: the mental association of heard sounds with *just* intervals (the intervals whose frequencies are related by simple, whole-number ratios) including their harmonic meaning and root implications. Tone representation is based on Hugo Riemann's idea of *Tonvorstellung,* as well as recent work by the composer and theorist James Tenney. The first chapter of this dissertation establishes the five basic premises of my argument. Where possible, I have offered supporting evidence from related disciplines, including acoustics, psychoacoustics, and music psychology. Each premise is illustrated with one or more musical examples.

Chapter 2, "A Theory of Tone Representation," lays out the technical apparatus of my theory. After explaining some of the basic tools of ratio intervals, I offer three basic preference rules for analyzing music using the principle of tone representation. These preference rules are based on the theory that in our cognitive processing of music, we tend to prefer the *simplest* harmonic explanation of any given collection of sounds—we prefer to minimize the number of fundamentals or harmonic roots, to posit the simplest just ratio relationships (i.e. the ratios expressible with the smallest numbers), and to avoid drastic "mental correction" of out-of-tune intervals. What emerges is not a strict algorithm for analysis, but rather a flexible set of preference rules and analytical techniques which can be applied pragmatically depending on the musical context.

In the third chapter, "Extended Just Intonation in Theory and Practice," I examine various twentieth-century applications of the ratio model of interval, with an emphasis on the expansion of just intonation to include higher prime numbers than the 2s, 3s, and 5s of Renaissance tuning theory. Beginning in the early-twentieth century, mainstream composer-theorists like Arnold Schoenberg and Paul Hindemith often evoked the acoustical properties of just intervals as an explanation for complex twelve-tone harmonies: later, precisely tuned extended just intonation was advocated by experimental American composers Harry Partch and Ben Johnston. European composers interested in just intervals and the closely related overtone series include György Ligeti and the "spectralists" Gérard Grisey and Tristan Murail.[1] In this chapter, the different theories and aesthetics of these composers are compared, with brief analyses of selected works.

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[1] The simple ratios between frequencies that define just intonation are also found between the pitches of the overtone series. The overtone series (also "harmonic series") is a collection of frequencies in the series f, $2f$, $3f$, $4f$, $5f$, etc. (where f is the *fundamental* frequency). Between the third and fourth members of the series (to take an example), we find the just interval 4/3, a perfect fourth. By extending the series indefinitely, we can find all just interval ratios between members of the series.

The idealized frequency relationships of the overtone series are closely approximated in the harmonic spectra of many real-world sounds. Most sources of sound produce a combination of vibrations at *many* frequencies: one of the most remarkable properties of our auditory system is its ability to accurately recognize that these frequencies come from a single source, and to combine the frequencies into a single perceived sonic object. The harmonic spectrum of a sound describes the specific frequencies and amplitudes of all of its component periodic vibrations. Any of these component vibrations can be called a *partial.* While (by definition) the frequencies of overtones always follow the harmonic series f, $2f$, $3f$, etc., a sound may have partials that are *inharmonic:* for example, the sound of a bell includes many partials which do not fit into any one harmonic series. When a sound's partials *do* match the harmonic series, they will be identical with overtones: we can refer to them as *harmonic partials.* The physics and psychoacoustics of the overtone series and just intervals are discussed in more detail on pages 20-25 below.

The fourth and final chapter examines music by the French spectralist composer Gérard Grisey (1946-1998). Spectralist composers use the harmonic spectra of natural sounds as basic musical material—for example, the beginning of Grisey's piece *Partiels* imitates the spectrum of a trombone's low *E* by assigning each partial of the sound to a member of the instrumental ensemble. Grisey often used distorted versions of harmonic spectra as a contrast to the pure tunings of the overtone series—such distortions raise interesting questions about the way we perceive harmony, which the theory of tone representation can help to answer. This chapter includes a brief introduction to the spectral techniques developed by Grisey and an analysis of his complex late work *Vortex Temporum* (1994-96).

CHAPTER 1: Basic Premises

Introduction: Interval as distance, interval as ratio

Interval, the relationship between two pitches, is the most basic concept of Western music theory—it is not from any single pitch, but from the combination and comparison of pitches with one another that musical meaning begins to emerge. Though the concept of interval is essential to our thinking about music, it is far from straightforward—the sense in which we use the term today combines ideas from different periods of musical history into a complex hybrid of different musical practices and mathematical abstractions.

Throughout the history of Western music, theorists have proposed two fundamentally different models of interval: one is based on the idea that interval quality is a result of the *ratio* between the frequencies of two pitches, while the other defines interval as the *distance* in pitch height between the two pitches.[1] Simple frequency ratios like 2/1 (the octave) and 3/2 (the perfect fifth) produce the perfectly tuned harmonic intervals which are essential to our music theories, though these intervals are often described as distances instead: twelve semitones or seven diatonic scale steps for the octave, seven semitones or four scale steps for the fifth. The intervals with simple frequency ratios are closely related to the physical properties of vibrating bodies—these

[1] Several music theorists have described this split between two different ways of understanding pitch. Ben Johnston explains the split as the difference between two different types of psychological organization, one based on an "interval scale" and the other on a "ratio scale"—this is based on the work of psychologist S. S. Stevens: see Johnston, "Scalar Order as a Compositional Resource," *Perspectives of New Music* 2/2 (1964): 56-76. James Tenney makes a similar distinction when he describes "harmony" as "the aspect of music and musical perception that has to do with relations between pitches other than simply the relations higher/lower or up/down" (*American Mavericks* interview, http://musicmavericks.publicradio.org/listening, accessed April 15, 2008). The distance conception of interval is the basis of several influential spatial models in recent theory including Neo-Riemannian transformational theory and the geometrical music theory of Dmitri Tymoczko, Clifton Callender, and Ian Quinn.

intervals can be found between the overtones of a complex periodic sound like a violin or flute tone. In broad terms, the ratio model tells us about the sonic quality (the degree of concordance and stability) and the root implications of an interval, while the distance model offers an easy way to measure and compare intervals, but ignores the way that the constituent pitches combine acoustically.

Both models have an impressive pedigree—the discovery of the correspondence between integer ratios and the basic musical consonances is attributed to the Greek philosopher Pythagoras in the sixth century BCE,[2] and a distance theory was proposed by Aristoxenus of Tarentum about two hundred years later. Aristoxenus rejected the Pythagorean ratio model in favor of a distance model based on the equal divisions of the tone into two, three, or four parts as units of measurement (such *equal* divisions of the 9/8 whole tone were an impossibility in Pythagorean theory). Aristoxenus found the then-prevalent Pythagorean theories too dependent on numerical abstraction and out of touch with musical practice: as Annie Bélis describes his innovation, "No longer could music be a matter of calculating intervals expressed by the relationship of two numbers, for its concern is not mathematical entities but sound..."[3] For Aristoxenus, melodic intervals had to be explained in terms of melody and sound itself, not as the result of abstract numerical relationships. He conceived of "pitch as a linear dimension on which notes appeared as points, rejecting the Pythagorean treatment of notes as quantities, and

[2] In Greek theory, ratios were measured between string lengths on a monochord, not between frequencies (the correlation of frequency with pitch was not discovered until the sixteenth century). String lengths are inversely related to frequency: since longer strings yield lower pitches with lower frequencies, strings with lengths in the ratio 5/4 produce frequencies in the ratio 4/5.

[3] Annie Bélis, "Aristoxenus," *Grove Music Online,* http://www.grovemusic.com, accessed April 15, 2008.

intervals as ratios between them."[4] His treatise, written entirely without the use of ratios, nonetheless relies on the existence of intervals like the fifth, fourth, and whole tone—the intervals that Pythagoreans defined by the ratios 3/2, 4/3, and 9/8. With his stated aim of avoiding ratio descriptions of these intervals, Aristoxenus must instead define them as givens of auditory perception—we agree "by ear" that the intervals of the octave, fifth, and fourth are consonances. The Aristoxenean approach is about *describing* an existing practice, while Pythagorean approaches try to explain why each interval by an appeal to eternal truths of number and proportion.[5]

The schism between Pythagorean and Aristoxenean viewpoints is described by Lawrence Zbikowski as the distinction between two different "cross-domain" cognitive mappings: Pythagoras invokes the idea of pitches as physical objects with fixed numerical properties, while Aristoxenus proposes "a mapping from the familiar domain of two-dimensional space onto that of music." As Zbikowski summarizes, "On closer inspection, the Pythagorean and Aristoxenean construals of interval are indeed incommensurate. From the Pythagorean perspective, pitches are physical objects, and an

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[4] Andrew Barker, *Greek Musical Writings,* vol. 2 (Cambridge: Cambridge University Press, 1989): 123-124.

[5] Traditionally, Aristoxenus has been associated with the evidence of the senses *(sensus),* and Pythagoras with numerical measurement *(ratio).* More recent developments complicate this picture. Since the nineteenth-century research of Hermann von Helmholtz, it has been apparent that what lends the intervals that Aristoxenus *sensed* as stable their stability is precisely the *rational* relationship between the frequencies of their constituent pitches. Simple ratios between the two pitches allow their overtones to align in a way that gives the sensory effect of consonance. Thus, in a reversal of the traditional view, the ratios of Pythagorean theory (supplemented by research in auditory perception after Helmholtz) can tell us a great deal about the sensual properties of intervals.

In recent theoretical work, the distance model pioneered by Aristoxenus has been far more prominent than the ratio model, but it is often applied in an abstract way that is distanced from the concept of *sensus.* Many recent distance-based theories use the concept of interval as distance primarily as a numerical measurement, not a reflection of aural experience. In such theories, we see Aristoxenus's distance approach to interval allied to the kind of mathematical abstraction typically associated with Pythagoreanism. The theory of tone representation developed in this dissertation attempts to balance *ratio* and *sensus* by drawing on both Pythagorean and Aristoxenean ideas: the theory invokes interval ratios, but in a way that reflects sensory experience (as described by modern psychoacoustical research).

interval describes the relationship between these objects. From the Aristoxenean perspective, pitches are breadthless points that simply mark out an expanse of two-dimensional space, and an interval is the expanse itself. Each mapping gives an account of interval, but each leads to a different conceptualization of musical structure."[6] The two models remain in an uneasy coexistence today, though familiarity has made it easy to overlook the deep-rooted contradictions between the two concepts; the tension between distance and ratio is apparent in definitions such as this one (from a Harvard University course website): "An interval is the *distance,* or difference in pitch height, between two notes. It is also the *sound* of two pitches occurring at that given distance" (emphasis added).[7]

An interval, then, is both a "distance" and a "sound"—and this strange double nature also affects the terms we use to describe intervals. We often emphasize the distance aspect of interval, with terms like "augmented fourth," "semitone," or "interval class 6"; in other situations, we're more interested in the sonic quality of the interval: "dissonant," "smooth," "stable," "beating." In day-to-day usage, the overlap between distance and sound doesn't present a problem—we happily accept that the interval of a perfect fifth has properties of both types: distance (defined as "seven semitones," "a major third plus a minor third,") and sound ("a perfect consonance," "ringing"). When it comes to building a more formal music theory vocabulary, though, confusions between the two aspects of interval can become problematic: as for example when we find

[6] Lawrence Zbikowski, *Conceptualizing Music: Cognitive Structure, Theory, and Analysis* (New York: Oxford University Press, 2002): 5-17.

[7] Website for First Nights (LAB-51), taught by Thomas F. Kelly. Theory Tutorial. http://isites.harvard.edu/icb/icb.do?keyword=k16235, accessed April 15, 2008.

ourselves speaking of a *distance* as consonant or dissonant, rather than the *sound* of the two pitches together.

Some theorists have sidestepped the problem by allowing *only* the distance aspect of interval into their formal theories: pitch-class-set theory, for example, simply omits the sonic quality of intervals (with the exception of the octave) from its model of pitch relations.[8] As a result, pitch-class theories excel at describing distance-based musical geometries—motivic transformations, symmetries in pitch and pitch-class spaces, etc.—but the relationship of these abstract geometries to any actual sonic properties is not explicitly theorized. Pitch-class set theory defines any interval by the distance between its constituents measured in equal-temperament semitones: for example, the major third comprises four semitones. To say anything about the sonic qualities of this interval, though, pitch-class set theory needs to be supplemented by acoustic facts—the interval is usable as a consonance and implies root status for its lower note only because it closely approximates the just intonation major third 5/4. Such qualitative information is only available by turning away from pure pitch-class-set thinking into a body of theory which is informed by psychoacoustics—a body of theory which takes the ratio, rather than the distance approach to interval.[9]

Outside the mainstream of academic music theory, theorist-composers like Harry Partch and La Monte Young developed complex harmonic systems based entirely on the

[8] Robert Morris describes the equally-tempered space of pitch-class-set theory as "fundamentally different from pitch-spaces where intervals have been traditionally produced from ratios of integers." See Morris, *Composition with Pitch-Classes: A Theory of Compositional Design* (New Haven, Conn.: Yale University Press, 1987): 35.

[9] It should be remembered that all musical temperaments are compromises based on an ideal of just intonation—all the intervals in our chromatic scale have a counterpart in just intonation, and in fact the equal-tempered scale itself evolved historically as a way to approximate many just intonation relationships. Pitch-class-set theory takes the equal-tempered scale as a given, ignoring the acoustical reasons for its historical development.

ratio approach to interval. These systems are more cognizant than distance-based theories of the acoustical quality of each interval, but demand extremely precise tuning—to a degree which can be impractical for musicians to realize in performance except on specially designed or modified instruments (Partch's homemade orchestra, for example, or Young's "well-tuned piano"). For these composers, purity of intonation is paramount—in many cases, this focus on interval quality leads to a drastically minimalist approach, as in Young's sine-wave installations or certain works by James Tenney. In these works, the long sustain of perfectly tuned just sonorities allows a meditation on sonic quality, but precludes the melodic and motivic development better explained by distance models of pitch.

In this dissertation, I seek a rapprochement of the two concepts of interval. In the context of current mainstream academic music theory, which has strongly favored the distance model, this rapprochement will entail the greater inclusion of ratio-based thinking. Readers whose background is in the compositional theory of extended just intonation may find that my argument does the opposite, injecting distance-theory characteristics into the pure just intonation model to make it more flexible and practical for analytical work. To me, it is undeniable that both distance and ratio play important roles in our musical experience—Western tonality is a historical instance of a musical language which strikes a careful balance between distance and ratio to create structures of extraordinary complexity and cognitive efficiency.[10] In the chapters which follow, I explore ways that this productive balance between the two modes of thought could inform the analysis of contemporary music.

[10] An engaging discussion of the balance of ratio and distance in tonal music can be found in Andrew Mead's article "Bodily Hearing: Physiological Metaphors and Musical Understanding," *Journal of Music Theory* 43/1 (Spring 1999), 1-19. This article is discussed in greater detail later in this chapter.

Accepting a different approach to interval can lead to a substantial rethinking of our entire theoretical apparatus. In recent years, one of the most influential reformulations of the idea of interval has been David Lewin's advocacy of transformation in place of distance.[11] By reintroducing ratios to this discussion, we find another language for thinking about relationships. What if we think of interval not as a distance (Forte et al.) or a transformation (Lewin), but instead as a kind of *attunement*? When we sing a just interval with another singer, our experience is not of measuring a distance from the other pitch, but of fitting our voice into our partner's sound to create a single, blended sonic quality. Under the ratio model, we can understand interval not as a distance or a path to be traversed, but rather as two becoming one—two pitches becoming a single interval, a single quality. By describing intervals as just ratios rather than distances in an abstract space, we can speak of the sensual aspects of interval instead of the more cerebral measurements offered by the distance approach.

Even as academic theorists have focused more and more narrowly on a distance model of interval, contemporary composers (including György Ligeti, Alvin Lucier, Gérard Grisey, and many others) have been increasingly drawn to sonic phenomena better explained by the ratio model. The move away from serialism as a dominant aesthetic has led some composers toward a new interest in sound as a physical phenomenon: "sound as sound" as opposed to the abstractions of serial intervallic

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[11] See Lewin's *Generalized Musical Intervals and Transformations* (New Haven, Conn.: Yale University Press, 1987). In an earlier article ("On Generalized Intervals and Transformations," *Journal of Music Theory* 24/2 (Autumn 1980), 243-251), Lewin describes the difference between distance and transformation—the system he describes "starts with the notion of transforming its objects, one into another, and then defines, as the interval from *s* to *t,* a certain transformation, unique of its sort, which carries *s* to *t.* By this means, transposition operations are actually conceived as defining intervals, rather than vice versa. [...] It is often useful to think of an interval *i* not as an abstract directed "distance" from *s* to *t,* but rather as a label for the corresponding transposition operation *Ti,* a unique operation of its kind which 'moves *s* to *t.*'"

structure. Such composers have proposed a variety of compositional theories—from pure extended just intonation in the music of Ben Johnston to the approximations of exotic sound spectra by Tristan Murail—but to date there have been few attempts to offer a theory of *listening* for this repertoire. Such a theory would apply equally well to music conceived under very different aesthetic ideologies, foregrounding the commonalities in our listening experience from work to work.

A ratio-based theory of listening will often entail the inclusion of higher numbers and more complex ratios than one finds in conventional just intonation theory; Ben Johnston refers to the tuning systems including higher prime numbers like 7, 11, and 13 as "extended just intonation." The idea that new music requires new and more complex intervals echoes Arnold Schoenberg's belief that the history of harmony could be described as a continuous climb into higher regions of the overtone series.[12] Thus, the dissonant harmonies of many twentieth-century works can be understood as an *extension* of earlier harmonic practices to include more distant harmonic ratios, rather than a complete abandonment of earlier harmonic practice, as suggested by the word "atonal." (Schoenberg disliked the description of his own music as atonal, and suggested that "pantonal" would be more accurate—though this never caught on among musicians or musicologists.)

If the ratio model, extended to the higher primes, accurately reflects how we perceive musical harmonies, it can offer an *interopus* theory of harmony applicable to music of many different styles and eras. Because the intervals built from the prime numbers 7, 11, and 13 fall "between the keys" of standard twelve-tone equal temperament, the ratio model is especially applicable to music that uses microtonal

[12] Arnold Schoenberg, *Theory of Harmony* (Berkeley: University of California Press, 1983): 21.

intervals. However, the ratio model of interval can also shed light on non-microtonal music, including many "atonal" works written in standard equal temperament. My model of interval offers an alternative to the pitch-set-class analysis which is usually applied to this repertoire—understanding intervals as ratios makes it possible to draw convincing links (based on shared harmonic roots) between groups of pitches belonging to different set classes or of different cardinalities.

Ratios offer a way of comparing harmonies across repertoires that superficially might seem very different—chords from atonal piano pieces by Schoenberg and Scriabin can stand next to the sonic installations of Alvin Lucier and La Monte Young or the just intonation works of Harry Partch. Concert-hall music since the 1960s has often drawn inspiration from acoustics and just intervals, as part of an aesthetic reaction against the abstractions of serial music. The ratio-based model of pitch is particularly suited to this repertoire, which includes (but is not limited to) the music of American experimental composers like Johnston, Tenney, Lucier, and Young, French spectralist music by Murail and Grisey, and works by Ligeti, Stockhausen, and Scelsi. While some scholars of twentieth-century music (especially Milton Babbitt) have emphasized the "contextuality" of twentieth-century works—the way each work creates its own internal structural expectations and coherence—my interopus approach attempts to describe some of the listening mechanisms that we bring as listeners to every work we experience. The theory I offer here can make similar statements about music in all kinds of tuning systems and pitch spaces, since it is conceived from the position of a listener, not as an abstract, work-specific mathematical system.

My motivation for promoting the ratio model is essentially *pragmatic*—this model offers a way of thinking about and discussing aspects of music which are difficult or impossible to describe with theories and vocabulary based on the distance model. The pragmatic method, as described by William James, tries "to interpret each notion by tracing its respective practical consequences."[13] Paraphrasing Charles Peirce, James continues: "to develop a thought's meaning, we need only determine what conduct it is fitted to produce: that conduct is for us its sole significance." A music theory based on pragmatism is thus *instrumental*—its goal is to further our understanding of our experience of the world, not to get at some absolute, abstract truth. James also proposes an instrumental view of truth: "ideas (which themselves are but parts of our experience) become true just in so far as they help us to get into satisfactory relation with other parts of our experience." When we take a pragmatic approach to theory, truth is not determined by the mathematical elegance or even internal logic of a system, but rather by the ability of the theory to convincingly make sense of musical experience. By taking musical experience seriously rather than building abstractions in a vacuum, a pragmatic music theory stands or falls by the musical value of the thoughts and further experiences that such a theory makes possible.[14]

The pragmatic theory that I propose in this dissertation is valuable, then, to the extent that it is useful—as Schoenberg wrote in his *Theory of Harmony,* "whenever I theorize, it is less important whether these theories be right than whether they be useful as

[13] William James, "What Pragmatism Means" in *Pragmatism, a New Name for Some Old Ways of Thinking: Popular Lectures on Philosophy* (New York and London: Longmans, Green, and Co., 1922): 2.

[14] The suspicion of abstractions in a pragmatic theory echoes the distinction drawn by D'Alembert between the *esprit systématique* and the *esprit de système.* See Jean Le Rond D'Alembert, trans. Richard N Schwab, *Preliminary Discourse to the Encyclopedia of Diderot.* (Chicago: University of Chicago Press, 1995): 22, 95.

comparisons to clarify the object and to give the study perspective."[15] I agree with Schoenberg that the strength of musical theories depends on how well they allow interesting or convincing discussion of pieces of music—applicability in a good close reading is more important than theoretical elegance or formal completeness. A pragmatic music theory can affirm multiplicity as a positive trait, rather than condemning it as the lack of unity—it values analyses which reflect the complexity of experience over those which flatten experience into a single, "unified" description. The ratio model can clarify certain aspects of the musical "objects" we investigate, though I do not see it as completely displacing established distance-model theories—rather, it can fruitfully be applied in combination with other models of pitch.

The idea of interval ratios may seem abstract and far-removed from the direct aural experience of listening to music. I would argue, however, that interval ratios are the key to explaining some of our most visceral musical intuitions and offer a way of formalizing and clarifying those intuitions in a diverse range of musical contexts. Taken as a vivid description of our experience of sonic quality and stability, interval ratios are not an obsolete remnant of historical music theory, but remain deeply relevant to the way we make and hear music today. In this dissertation, I advocate a ratio model of interval (drawing on the theories of Hugo Riemann and James Tenney) which allows a degree of *tolerance* or mistuning. Psychoacousticians and music psychologists have experimentally confirmed the importance of frequency ratios to the way we hear and understand pitch

[15] Schoenberg, *Theory of Harmony,* 19; quoted in Cook, "Epistemologies of Music Theory," in *The Cambridge History of Western Music Theory,* ed. Thomas Christensen (Cambridge: Cambridge University Press, 2002): 96. Cook's article includes a detailed exploration of the explanatory goals of music theory. In a pragmatic approach, "right and wrong" don't apply to music-theoretical ideas (except in the trivial cases of self-contradiction)—rather, music theories (and analyses) succeed or fail based on how well they convince us of their results, how much they intrigue and fascinate us, and whether they help us to a deeper musical understanding.

combinations. Such experiments suggest that even out-of-tune versions of just (ratio-based) intervals can carry the same musical meanings as the perfectly-tuned versions.[16] Hugo Riemann's early-twentieth-century theory of *Tonvorstellungen* (tone representations) made a similar proposition: that the harmonic sense we make of all heard intervals depends on our understanding them as "representations" of purely-tuned just intervals. Accepting this degree of tolerance helps to free ratio-based pitch theory from its Pythagorean, numerological heritage, and suggests broader analytical applications.[17] This model can provide the vocabulary and tools we need to analyze an important body of contemporary music which is poorly served by existing, distance-based theories—and also insufficiently explained by a "purist" just intonation approach. This model is the basis for new analytical tools, the usefulness of which I will demonstrate in analyses of music by Ligeti, Grisey, and others.

The acceptance of a degree of mistuning is a pragmatic decision—this tolerance for deviation keeps us in touch with the intuitions of our surprisingly forgiving and imaginative ears, and avoids the arguments over very small pitch differences which can arise from a "purist" just intonation approach. The broader scope for interpretation allowed by tolerance broadens the range of works which can be illuminated by ratio theory, and also recognizes the complexity and messiness of our experience of music in time—even the simplest harmonic progressions can demand many reinterpretations of the meaning of pitches and intervals as they unfold.

[16] For an overview of psychological research on tuning and temperament, see E. M. Burns and W. D. Ward, "Intervals, Scales, and Tuning," *The Psychology of Music,* second edition, ed. Diana Deutsch (San Diego, California: Academic Press, 1999).

[17] This is another instance of balancing the demands of *ratio* and *sensus:* as in Pythagoreanism, just ratios provide the essential points of harmonic reference in my theory, but the Pythagorean demand for absolutely precise tuning is relaxed (based on psychological research into aural perception) to include approximate versions of the referential just intervals.

As a preliminary to the full theory of tone representation for twentieth-century and contemporary music presented in Chapter 2, this chapter establishes the basic premises of the theory, illustrating each premise with musical examples. These range from piano pieces by Scriabin to Tuvan throat singing and the microtonal improvisations of La Monte Young. The five premises which I propose are listed below—detailed explanations and demonstrations of each follow in the remainder of this chapter.

1. Just intervals are the referential intervals for harmonic perception.[18]
2. The principles of standard just intonation (based on the integers 2, 3, and 5 and their multiples) can be extended to include higher prime numbers.
3. Each just interval implies a fundamental or root, and a specific closely-related harmonic domain based on the overtones of that fundamental.
4. We recognize just intervals even when they are slightly mistuned.
5. Faced with large and complex harmonies, we tend to resolve them into combinations of multiple simpler harmonies if possible.

The brief discussions of musical works in this chapter are not intended to be exhaustive analyses, but rather introductory demonstrations of certain important concepts which will be developed in the remainder of this dissertation: e.g., intervals as frequency ratios, the natural overtones of harmonic sounds, and the cognitive recognition of

[18] The term "referential" is borrowed from the theoretical writings of James Tenney, whose assertion that the simple just ratios are referential for the auditory system I have adopted here. We understand intervals near any of the simple just intervals as variants of those intervals; the heard interval is takes on a harmonic meaning by *reference* to the just interval it approximates. James Tenney, "The Several Dimensions of Pitch," in *The Ratio Book: A Documentation of the Ratio Symposium, Royal Conservatory, The Hague, 14-16 December 1992,* ed. Clarence Barlow, Feedback Papers 43. (Cologne: Feedback Studio Verlag, 2001): 102-115: 110.

approximate versions of just intervals. This survey emphasizes the work of American experimental composers including Alvin Lucier and James Tenney, in large part because their minimalist works explicitly address many of the pitch phenomena I plan to discuss. Aspects of pitch which are foregrounded in the music of Tenney and Lucier play a similar, if less obvious, role in music of many styles. As noted above, one of my guiding assumptions in this survey is that we bring many of the same listening strategies to our experience of different kinds of music—that we don't completely exchange our "tonal" ears for "atonal" ones when we switch from Mozart to Schoenberg, or our "experimental" ears for "serial" ears when we switch from Lucier to Boulez. Thus, we can take insights from the complex microtonal pitch world of Tenney and Lucier, and apply them even to works written for normally tuned instruments: though specifics may differ, the essential perceptual mechanisms remain the same.[19]

Premise 1: Just intervals are the referential intervals for harmonic perception.

James Tenney: *Koan* for solo violin (1971)

Let's begin with a riddle. In Zen Buddhism, a *koan* is a brief story or question which serves as the focus of a meditation—*koan*s often seem paradoxical or meaningless, but are intended to direct the Zen student to sudden and illuminating shifts in perspective,

[19] This is not to say that the importance of the distance and ratio models are equal for all musical situations. A motivically dense composition by Schoenberg, for example, might demand more from our "distance" model of interval, while a microtonal piece by Tenney will depend much more on hearing the interval qualities and harmonic implications defined by ratio. Tonal music draws on both aspects of interval, which is perhaps one of the reasons for its expressive power and historical success.

breaking established patterns of thought. (A well known example is the *koan* “What is the sound of one hand clapping?” attributed to the Zen teacher Hakuin.[20])

James Tenney’s *Koan* for solo violin (1971) explores the surprising complexities of a continuous progression through the pitch continuum. The piece is one of a series of ten “postal pieces” written by Tenney between 1965 and 1971, each exploring a particular musical idea through a minimally-notated, postcard-sized score. *Koan* combines a continuous pitch glissando with the fixed pitches of the violin’s open strings. The violinist plays a “fairly slow tremolo,” rocking the bow from one string to the next. Beginning with the open fifth between the G and D strings, the violinist slowly slides up the G string toward the D above it. Compared to standard rates of pitch change in Western music, the speed of the glissando is glacial; in his recording, Marc Sabat takes about four minutes to span the fifth between G and D—this means that it takes about 35 seconds just to traverse a single semitone. After the violinist reaches a unison between the stopped G string and the open D string, he continues to slide upwards on the G string, expanding the interval from a unison to the fifth D-A. When he reaches the A on the G string, it’s possible to cross over to the open D and A strings with no change in pitch, and the whole process can start again with a glissando from D up toward A. This process continues into the upper reaches of the violin’s range. During the continuous upward glissando, the interval between the strings fans in and out between a fifth and a unison, finally opening out to an octave. The complete score of *Koan* is reproduced in Figure 1.1.

[20] Kamil V. Zvelebil, “The Sound of the One Hand,” *Journal of the American Oriental Society*, 107/1 (1987): 125-126.

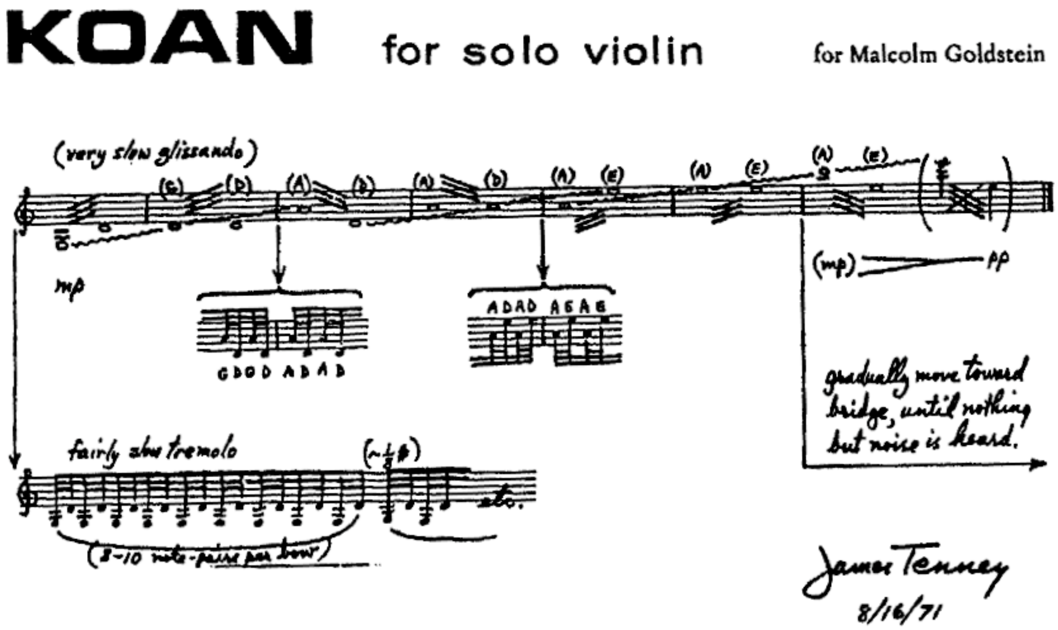

Figure 1.1: Score of James Tenney's *Koan* for solo violin. Copyright 1971 by Sonic Art Editions. Used by permission of Smith Publications, 2617 Gwynndale Ave., Baltimore, Maryland.

The goal of a *koan* is not a specific answer so much as a re-assessment of the very assumptions of the question. Similarly, Tenney's *Koan* seems to ask us to set aside our customary ways of thinking about pitch—the sparseness of musical events focuses our attention to the tiny changes in intervallic quality. *Koan* baffles our analytical tools, and in a sense, the very practice of analysis and close reading. On one hand, *Koan* is too obvious—the structure expressed in the score is a complete description (at least at the global level) of the work. It would seem that there's no "hidden structure" here, waiting for explication by some eager analyst.[21] On the other hand, *Koan* gives us little that our

[21] See Ian Quinn's discussion of the analytical difficulties posed by minimalism in "Minimal Challenges: Process Music and the Uses of Formalist Analysis," *Contemporary Music Review* 25/3 (June 2006): 283-294. An analysis based solely on pitch distance would have little to say about *Koan*—such an analysis might accurately describe the pattern of changing intervals, but it's difficult to see how that would bring anything to our attention that isn't already implicit in the score. Like many of Tenney's other Postal Pieces, *Koan* is in part a critique of Western notation practice—in all these pieces, much more happens sonically than the notation itself indicates. For example, in *Beast* (1971), Tenney graphically notates the

standard analytical tools can grasp—except for the pivotal open-string fifths and unisons, there are no clearly notated intervals, just a gradual slide from one fifth to the next. What could there be to say about the slowly evolving relationship between pitches?

The sense of any independent "notes" is obliterated by the continuity of the sound and the gradual changes in pitch. In a recent interview, Tenney describes his fascination with glissandi:

> I think one of the things that the glissando does is to remind us that frequency is a continuum and we don't have to think of it in scale steps. We can organize it in a number of ways that break it down and structure it as something other than as a continuum, which I certainly do in any harmonic series piece. It's in a sense a reminder of a certain reality, a physical reality about that parameter.[22]

Koan is a reminder that despite the typical division of pitch space into discrete points, pitch is in fact continuous. The piece strips away the conventional signposts of interval size, leaving us confronted with the wilderness of the unbroken pitch continuum—in a sense then, we are in a "pre-theoretical" state, much as musical thinkers would have been before the theoretical breakthrough attributed to Pythagoras in the sixth century BCE.[23] Before Pythagoras's theoretical innovations, Greek music had an established practice with specific scales and intervals, but there was no explanation for why certain intervals

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changing beat frequency of a closely tuned double bass interval, and in *Having Never Written a Note for Percussion* (1971), he notates nothing more than a single note, with a crescendo followed by a diminuendo, and the notation "very long." We can distinguish between two roles for pitch notation: one, a "sound-picture" of what one hears, and two, a "tablature" for realizing the piece. In traditional notation, these are closely related—a score can be used both as tablature and as a sound-picture—but Tenney separates the two, with a simple "score" that offers complete instructions for realizing a very complex sonic result. If we want to discuss this sonic result, we need to turn to the sounding music, not just the score—and we soon find aspects of the sound which can only be approached through the ratio model of interval.

[22] James Tenney and Donnacha Dennehy, "Interview with James Tenney." *Contemporary Music Review* 27/1 (February 2008), 79-89: 89.

[23] While Pythagoras is traditionally credited with the discovery of the interval ratios, there is little concrete historical proof that he is responsible. Others responsible for this discovery, as Andrew Barker notes, might include Pythagoras's disciples or anonymous practical musicians and instrument makers. See Barker, op. cit., 28-45.

were chosen and not others. Pythagoras's theory offers an explanation, for the first time, of why certain intervals have musically desirable properties—stability, smoothness, etc.—while others do not.

* * *

As Nicomachus of Gerasa (active in the early second century CE) tells the story, before Pythagoras's revelation the size of intervals could be measured only "by ear"—there was no way to check the error-prone sense of hearing in the same way that a visual estimation of size could be checked with a measuring rod. Ptolemy (also writing in the second century) gives the example of two circles, one drawn "by eye" and the other drawn with a compass—the freely-drawn circle might seem accurate until it is compared to the mathematically precise circle, which is immediately recognizable as more correct.[24] In this pre-theoretical era, there was no way to rationally *measure* interval at all, except by aural approximation.

According to Nicomachus, Pythagoras was walking past a blacksmith's workshop when he heard the clanks of hammers on anvils ringing together in beautifully concordant combinations, including the intervals of the octave, fourth, and fifth used in Greek music. Inspired to investigate, he interrupted the blacksmiths and—after a "great variety of experiments" on the shapes of the hammers, the kind of iron being hammered, and even the strength of the blacksmiths—weighed each of their hammers. He discovered that the weights of the hammers which rang consonantly together were all in simple ratios to one

[24] Barker points out that Ptolemy's division between the senses and reason is complicated here by the fact that the senses are required to determine the superiority of the compass-drawn circle—to overcome this problem, Ptolemy offers a second premise, that "our senses are better equipped to judge such things than to construct them" (Barker, op. cit., 277, n9). The conflict between reason and the evidence of the senses is a common theme in music treatises of this period. See also Tenney's comparison of a geometrically ideal circle to the simple just intervals in his article "The Several Dimensions of Pitch."

another—their weights, from large to small, were precisely 12, 9, 8 and 6 units. The consonances which musicians had discovered by ear corresponded exactly with the ratios between the weights of the hammers: for example, the two highest hammers, with a weight ratio of 12/9 (simplified to 4/3), produced the interval of the perfect fourth.

Taken literally, the story of Pythagoras's discovery is clearly untrue—modern acousticians confirm that there is no direct correspondence between the weight of a hammer and the pitch of the sound it produces. (The hammer's pitch is determined by not just one, but a number of factors, including its density, material, and shape: its pitch is determined by our processing of a complex inharmonic spectrum, the sum of all its vibrating modes in three dimensions.) Such a direct correspondence does hold, though, between pitch and the lengths of string segments as measured on a monochord (the experimental instrument of the Greek music theorists)—we can best understand Nicomachus's story as an origin myth for the entire school of Pythagorean harmonists. With the monochord, theorists could compare the sound of a string vibrating in its entirety to the same string stopped halfway along its length, producing the interval of the octave with the ratio 2/1. A similar demonstration showed how the sound of the whole string compared to the sound of three-quarters of the string yielded the interval of the fourth, 4/3. The combination of intervals based on the numbers 1, 2, 3, and 4 gave rise to all the intervals of Greek theory. (Pythagoras and his followers saw this correspondence as a reflection of the numerological structure of the universe, as represented by the mystical *tetraktys*—the numbers one to four expressed as the rows of an equal-sided triangle.) The ratios of Pythagorean theory become increasingly complex when applied to intervals smaller than the whole tone, which need to be generated from repeated iterations of the basic intervals of 3/2 and 4/3; this iteration results in ratios like 256/243 (the *limma,* $(4/3)^5$) or 2,187/2,048 (the *apotome,* $(3/2)^7$).What has given the legend of Pythagoras and the smithy lasting interest is the way it dramatizes a strange discovery: that heard musical consonance can be explained by numbers—and not just any numbers, but by the simplest integers, 1, 2, 3, and 4. Somehow, the sensual quality of consonance is intimately tied to the abstract mathematical world of integers and their ratios.

* * *

Let's return to Tenney's *Koan*—in fact, let's focus on the first section of the glissando, the motion from the fifth G-D to the unison on D. What one immediately notices is that all intervals are not created equal—certain intervals seem to "click into place" at a specific moment in the glissando, while other intervals feel like they're in motion, sliding toward one of the more stable intervals.[25] It often seems like the pitches are straining toward the more stable points in the pitch domain—these stable intervals seem to exert a gravitational pull on nearby pitches. Figure 1.2 shows the points of stability that I hear in the opening of *Koan,* compared to the nearest simple just intervals. The most stable intervals, equivalent to standard tonal consonances, are shown in the top row—intervals that are only weakly stable are shown in the bottom row.

D	D	D	D	D
G	A	B♭	B	D
3/2	4/3	5/4	6/5	1/1
702¢	498¢	386¢	316¢	0¢

D	D	D
A♭	B♭↓	C
7/5	9/7	9/8
583¢	435¢	204¢

Figure 1.2: Stable intervals in *Koan* (less stable intervals in lower row)

[25] This typifies the psychological concept of "categorical perception," which is discussed further in relation to Premise 4 below.

Stable intervals in the glissando between G and D include the initial perfect fifth G-D, the perfect fourth A-D, the major third B-flat -D, and the minor third B-D. The stable intervals are precisely those which can be represented by simple frequency ratios: the fifth 3/2, the fourth 4/3, the major third 5/4, and the minor third 6/5. Even in this context, which is far removed from any traditional tonal syntax, these intervals have a quality of stability which differentiates them from the more mobile intervals in between.

The quality of stability is proportional to the simplicity of an interval's ratio—"perfect" consonances like the octave, fourth, and fifth have simpler ratios than the imperfect consonances of the major and minor thirds and sixths. The traditional consonances are not the only intervals in this excerpt which seem to act as intermediate goals within the overall motion—listening carefully to the spaces between the consonances, we can hear the same sort of "centering" on justly tuned whole tones (9/8) and even on unusual intervals invoking factors of 7 like 7/5 (a augmented fourth of about 583 cents, or hundredths of an equal-temperament semitone). As Schoenberg argued, the distinction between consonance and dissonance is relative, not absolute: there is a continuum from the most consonant intervals to the less consonant, from simple ratios to complex ones.[26] Where we draw the line is a matter of musical training and the degree of attention we pay to these intervals: violinist Marc Sabat has come up with an extensive list of just intonation "intervals tunable by ear," including such complex intervals as 9/7 (435 cents).[27]

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[26] Schoenberg, op. cit., 20-21.

[27] Marc Sabat, "Analysis of Tuneable Intervals on Violin and Cello (2004)," http://www.plainsound.org/, accessed April 15, 2008. Note that the degree of mistuning for just intervals that seems acceptable to a given listener seems to be at least in part a factor of musical training—the often heated arguments between proponents of different tuning systems often stem from the different degrees of tolerance to which each disputant has become accustomed.

By gradually changing the interval size, Tenney sets off our perception of interval as an object of aesthetic contemplation. In listening, we discover the irregularities in our reaction to intervals, and the strange "pull" that some pure-ratio intervals exert on neighboring pitches.

* * *

What is it, though, that makes the consonant intervals we hear in this excerpt of *Koan* correspond to the intervals with simple ratios? To the Pythagoreans, intervals with simple ratios were consonant because they reflected basic numerical and cosmological truths of the universe. Needless to say, this explanation has few adherents today, though we still need to explain the correspondence of simple ratios and perceived consonance: between *ratio* and *sensus*. The modern, psychoacoustical explanation is based on the way we hear the *overtones* of pitched sounds.

Until the Renaissance, interval ratios were always expressed as comparisons of string lengths on a monochord or comparable instrument. The discovery of the precise inverse relationship between string length and rate of vibration (usually attributed to Galileo Galilei), along with a new understanding of pitch as the perceptual correlate of periodic vibration rate, made it clear that the same ratios measured between string lengths also could be found between frequencies. The interval of the octave, for instance, could now be understood not only as the product of sounding strings with lengths in the ratio 2/1, but also as two vibrations in the air, one at twice the rate of the other. The shorter string (length 1, in the ratio 2/1) would have the fast vibration speed (let's say 2x), while the longer string (length 2) would have the slower vibration speed (1x). Vincenzo Galilei (Galileo's father) attacked the traditional equation of interval to specific ratios by showing how, in different experimental setups, basic intervals like the octave could be produced by ratios different from the standard 2/1. For example, if one compares the mass of the weights which held strings taut instead of comparing string lengths, the octave has the ratio 4/1, not 2/1.

Simple integer ratios between *frequencies* were discovered in a natural phenomenon (not just as the result of human musical practices) when Marin Mersenne described the lower reaches of the *overtone series* in his *Harmonie Universelle* (1636-37).[28] Mersenne found that while listening closely to instrumental sounds, he could discern more than one pitch—not just the nominal pitch of the sound, but higher tones arranged in a sort of chord above this basic pitch.[29] A scientific explanation soon followed: the vibrating body (for example, a string or the column of air in an organ pipe) vibrated not only along its entire length (yielding the lowest, or fundamental tone), but also (and simultaneously!) in halves, thirds, fourths, and so on. Because the frequencies of these pitches are all multiples of the same fundamental frequency, the intervals between them are always just intervals. In his *Harmonie Universelle,* Mersenne mused "it seems it is entirely necessary that [the string] beat the air five, four, three, and two times

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[28] René Descartes and Isaac Beeckman were important influences on Mersenne's thought, though their descriptions of the overtone series were not as sophisticated or complete as Mersenne's: see H. F. Cohen, *Quantifying Music: The Science of Music at the First Stage of the Scientific Revolution, 1580-1650* (Dordrecht: D. Reidel, 1984): 127, 167-172, 198.

[29] Modern scholars of psychoacoustics often refer to the distinction between *holistic* listening (hearing a collection of partials fused into a single complex timbre) and *analytical* listening ("hearing out" the individual partials as separate entities). Trained listeners can often switch back and forth between the two modes of listening. For a more detailed discussion, see William Sethares, *Tuning, Timbre, Spectrum, Scale,* second ed. (London: Springer-Verlag, 2005): 25-27.

in the same time"[30]—but he quickly (though incorrectly) dismissed this idea as physically impossible.

Mersenne's speculation was correct—though it would be a hundred years before scientists were able to prove it. These consecutive divisions of the vibrating body had their own vibration frequencies, each a multiple of the fundamental frequency—these are overtones, and collectively they are called the overtone series.

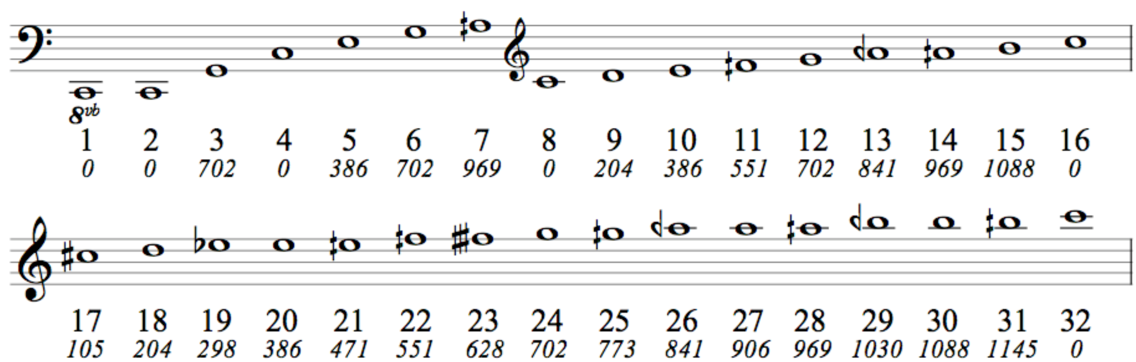

Figure 1.3: The overtone series on C, approximated to the nearest quartertone. Numbers in italics indicate the pitch class in cents, with C = 0.

For example, take a vibrating string tuned to 55 Hz (Hertz, or vibrations per second)—the pitch of a double bass's A string. In addition to the 55 Hz fundamental, the sound will contain an overtone at 110 Hz—at the interval of an octave (a 2/1 ratio) above the fundamental pitch. The next overtone, at 165 Hz, is an octave and a fifth above (3/1), followed by a double octave (4/1) at 220 Hz. The next overtone after this is in a 5/1 ratio with the fundamental, sounding two octaves and a just major third above: 275 Hz.

Despite Mersenne's doubts, a vibrating string can simultaneously vibrate in multiple *modes,* each corresponding to some division of the string into equal parts. The first part of the puzzle was solved by two English scientists, William Noble and Thomas

[30] Thomas Christensen, *Rameau and Musical Thought in the Enlightenment* (Cambridge and New York: Cambridge University Press, 1993): 136. I draw here on Christensen's research on the history of overtones and vibrational theory, particularly on the discussion in pages 133-168.

Pigot, who demonstrated the existence of nodal points—the points of stillness on a vibrating string—through an ingenious experiment. Noble and Pigot set up two strings, one long string, and one short string tuned an octave above the long string. When the higher string was plucked, sympathetic resonance would make the longer string sound as well—but the longer string would not vibrate along its whole length, but in two equal parts, with a still nodal point at the center. Noble and Pigot put paper riders along the long string—when it was set into motion, all of the riders would jump off except for those at the still point in the center of the string's length.[31]

Proof that a string could vibrate in different modes was not enough to show that such modes could coexist—Sauveur made the essential link between nodal points (which he discovered independently of the Englishmen) and the overtone series: he realized that the string can vibrate in several different ways at the same time. Bernoulli also contributed to the earliest theories of overtones—he seems to have anticipated Fourier's proof that sine tones in harmonic relation could compose any periodic vibration.[32] Sine tones of varying phase and amplitude can be combined to produce any kind of periodic waveform.

The sine tone components (partials) of a complex sound are not usually clearly audible as individual pitches—our auditory system tends to group them together into a single fused pitch entity. The relative intensity of the partials determines the timbre of the complex tone. Various expedients have been used to make the upper partials more audible—these include the resonators developed by Hermann von Helmholtz in the nineteenth century. Another way of "hearing out" these upper partials is to use a guide

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[31] Christensen, op. cit., 136.

[32] Christensen, op. cit., 150.

tone—for example, by first playing a harmonic on a guitar string (lightly touching a node so only an upper partial sounds) then plucking the string normally, one can hear the isolated harmonic as an independent entity within the complex sound.

Psychoacousticians explain consonance between complex tones (tones with many overtones or partials, as opposed to pure or sine tones) as the result of coinciding partials.[33] When the partials of two tones match in frequency, they blend with one another—but when they don't match, the adjacent partials interfere with one another, causing "beats" which, if powerful enough, can create a sensation of sonic "roughness." Matching of partial frequencies occurs when the fundamentals of harmonic tones are in a simple ratio relationship: for example, if the fundamentals are in a 3/2 ratio, the partials will coincide at 6, 12, 18, and so on (Figure 1.4).

2		4	**6**	8		10	**12**	14		16	**18**	
	3		**6**		9		**12**		15		**18**	etc...

Figure 1.4: Coinciding partials between complex tones a fifth (3/2) apart

Even if the ratio between the two pitches is not perfectly tuned, the combination of pitches can still be heard as consonant. As Albert Bregman explains, "near misses" between the partials create only mild beats which aren't perceived as discordant:

> In general, if the ratio between the two fundamentals is *m*:*n*, and *m* and *n* are integers, every *n*th harmonic of the first fundamental will correspond in frequency with every *m*th harmonic of the second. Therefore, when the ratio can be expressed in terms of small integers there will be many exact correspondences of frequencies. Even if the ratio is not exactly 2:1 or 3:2 or another such ratio, but is near these simple ratios, the partials that would have the same frequency in the exactly tuned case will be quite near to one another in frequency. Hence the frequency of the beats between

[33] This explanation originates with Hermann von Helmholtz: see Helmholtz, trans. Alexander Ellis, *On the Sensations of Tone.* New York: Dover, 1954 [1885]: 179-197. Consonance is discussed in most books on acoustics and psychoacoustics: for a particularly clear explication, see Brian Moore, *An Introduction to the Psychology of Hearing,* fifth edition (Amsterdam and Boston: Academic Press, 2003). For a comparison of different ideas of consonance and dissonance, James Tenney's *A History of "Consonance" and "Dissonance"* (New York: Excelsior Music Publishing, 1988) is a valuable resource.

> corresponding harmonics will be quite near zero and will not be unpleasant. Such near misses, rather than exact correspondences, occur frequently in the equally tempered scale of modern Western instruments.[34]

Currently, this is the most widely accepted explanation for musical consonance. The Pythagorean tradition in music theory has historically been associated with an abstract numerological cosmology: instead of relying (like Aristoxenus) on empirical evidence from listening (*sensus*), pride of place was given to purely mathematical definitions (*ratio*). Helmholtz's theory of consonance makes it possible to preserve the Pythagorean interval ratios on a much firmer empirical foundation based on the psychology of hearing. Ratios still play a preeminent role, but not because of numerology: rather, the simple ratios—when applied to the complex harmonic tones of musical instruments and voices—create the smallest degree of interference between partials of the two tones of an interval. Through psychoacoustics, we can move beyond the old dichotomy of *ratio* and *sensus:* abstract ratios between numbers are directly linked to the visceral sensations of consonance and dissonance. Psychoacousticians illustrate the relative consonance of intervals in a diagram called a "consonance curve"—the peaks of this curve (see Figure 1.5b) neatly match the intervals which we've identified as points of rest in *Koan.* [35]

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[34] Albert Bregman, *Auditory Scene Analysis* (Cambridge, Mass.: MIT Press, 1990): 505. The acceptance of such "near misses" as consonant is anathema to Pythagorean theories, which demand precise tuning of all ratios. Tolerance for approximation is more characteristic of an Aristoxenean, listening-based approach.

[35] An interesting question about the relation of consonance and the spectra of musical sounds is raised by the experiments of William Sethares, who uses stretched and compressed spectra to create "consonant" versions of traditionally dissonant intervals. By creating artificial tones in which each octave is stretched to a major ninth (with all other intervals adjusted similarly), Sethares creates situations where all of the partials of two tones coincide perfectly, though their fundamentals are separated by a "dissonant" minor ninth. Though Sethares's hybrids are fascinating, I find that for most musical situations the assumption of a harmonic spectrum for each tone is the most useful for analysis—this reflects the tendency of music theory to abstract "the note" away from real-world tones, but allows us to assume similar pitch relationships even when the scoring of a passage is quite different. See Sethares, op. cit.

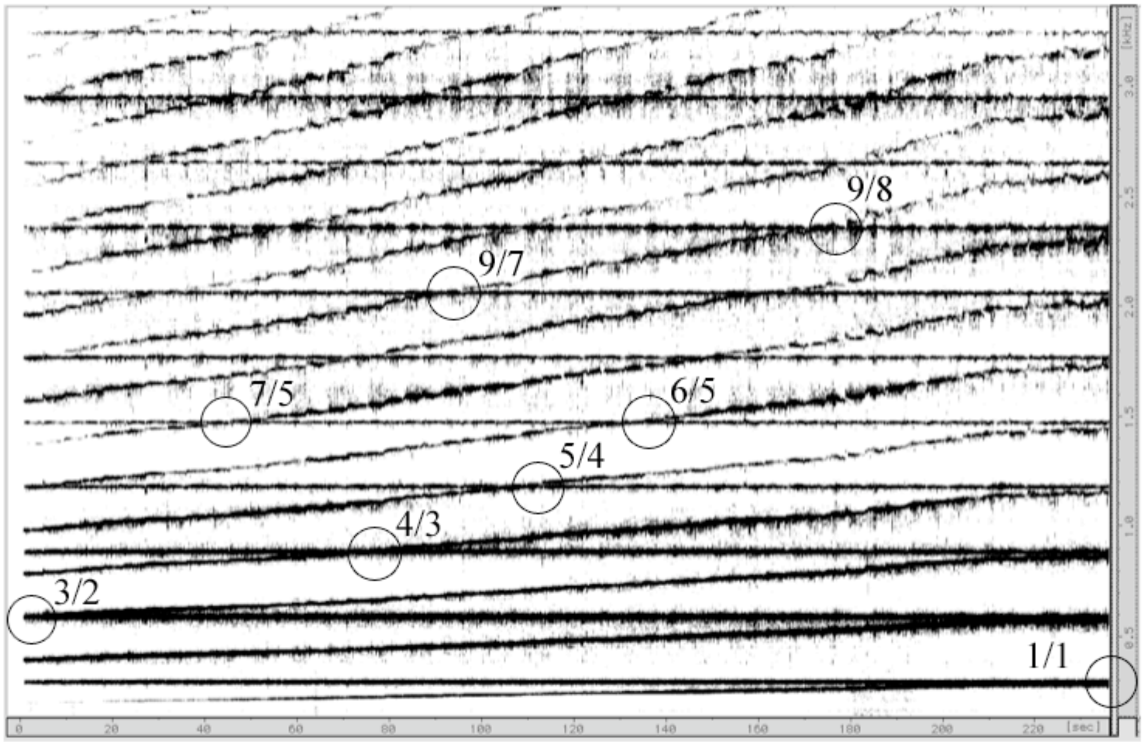

Figure 1.5a: Spectrogram of *Koan*

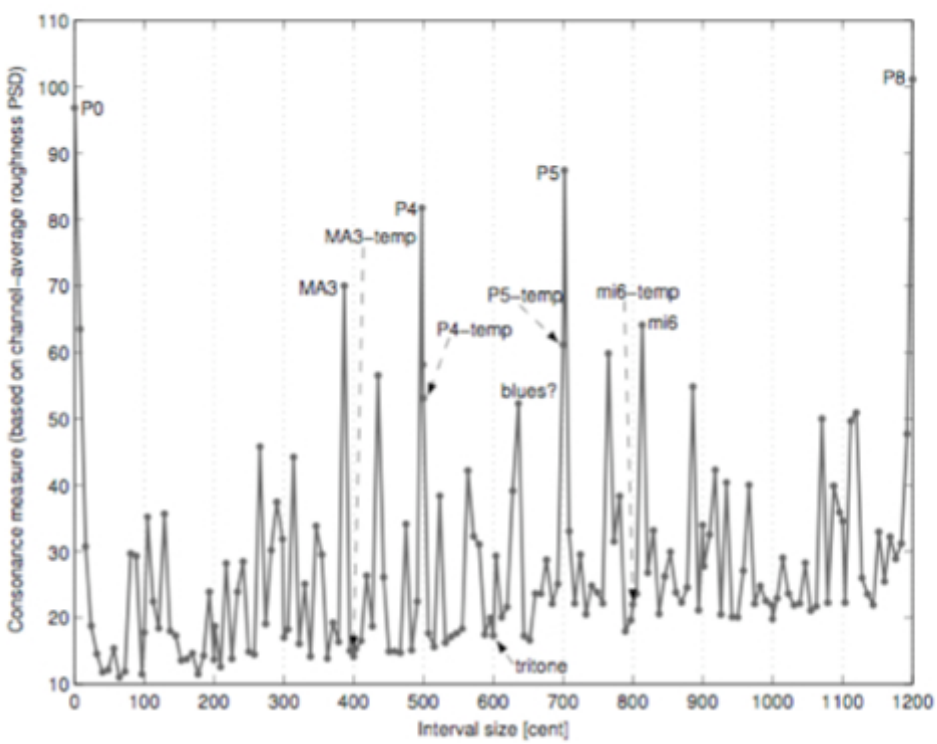

Figure 1.5b: Consonance curve (Skovenborg and Nielsen, 2002)

Figure 1.5a shows a spectrogram of the first four minutes of Marc Sabat's recording of *Koan*—this excerpt begins with the fifth G-D and ends just before the arrival of the unison on D. In a spectrogram of a sound, the sound's evolution in time is represented on the *x*-axis from left to right, and frequency is shown on the *y*-axis: low frequencies at the bottom, and high ones at the top. The intensity of vibrational energy at any frequency is indicated by shading from white to black: white means an absence of energy at that frequency, while black means strong energy. Shades of gray show intermediate gradations in intensity. On the spectrogram, we see not only the fundamental of each pitch, but also its partials: the harmonic partials of a complex tone appear on a spectrogram as a series of equally-spaced lines, reflecting the equal difference in frequency between them. In Figure 1.5a, the repeated pitch D and its partials are represented by horizontal lines, since the frequency does not change—the diagonal lines which cut across the D and its partials represent the glissando from G to D.

The just intervals which we hear as stable points in the glissando are the points in the spectrogram where the two sets of lines cross—for example, at the perfect fourth (ratio 4/3), the fourth partial of A matches the third partial of D. The interval at each crossing can be compared with the consonance curve taken from Skovenborg and Nielsen's model of consonance for harmonic tones (Figure 1.5b)—note the correspondence of the simple just intervals to the points of maximum consonance.[36] Again, as Schoenberg pointed out in his *Theory of Harmony,* the distinction between consonance and dissonance is relative, not absolute. The just intervals have varying

[36] Esben Skovenborg and Søren Nielsen, "Measuring Sensory Consonance by Auditory Modelling," *Proceedings of the 5th International Conference on Digital Audio Effects, Hamburg, Germany, September 26-28, 2002:* 251-256. See also R. Plomp and W. J. M. Levelt, "Tonal Consonance and Critical Bandwidth," *Journal of the Acoustical Society of America* 38 (1965): 548-560.

degrees of consonance, depending on the simplicity of their ratios—there's a high degree of correspondence between the partials of a perfect fifth (3/2), but fewer correspondences between the partials of a whole tone (9/8). By looking at this excerpt in terms of ratios instead of only as distances, we can begin to discuss these different degrees of consonance and their musical possibilities.

Within a continuum of interval sizes, intervals defined by simple ratios stand out as qualitatively different from the irrational or more complex intervals between them. In his theoretical writings, Tenney uses the word "harmony" in a specific sense to refer to just this sort of qualitative difference between intervals, which is separate from absolute interval size measured as distance; for Tenney, as we saw, harmony is specifically "the aspect of music and musical perception that has to do with relations between pitches other than simply the relations higher/lower or up/down."[37] These "other" relations draw on a cognitive tendency to group together partials that are all harmonics of the same fundamental: what Albert Bregman terms "harmonicity":

> There is a particular relation that can hold among a simultaneously present set of partials. They can all be harmonics of the same fundamental. There is a good deal of evidence to suggest that if they are, they will tend to be assigned to the same stream; that is, they will be fused and heard as a single sound. Let us call this the "harmonicity" principle. (232)

Though Tenney is concerned with pitches and Bregman with partials, many of the grouping mechanisms remain the same—for example, if the fundamentals of several complex pitches are themselves overtones of the same fundamental, all of the overtones of the complex pitches will also share that fundamental.

The harmonicity principle suggests that any approach to musical space that conceives only of distance relations—"higher/lower or up/down"—will misconstrue how

[37] Tenney, "American Mavericks" interview, op. cit.

we perceive pitches in combination with one another. In the field of visual cognition, Gestalt psychologists have described a "proximity principle": that "the closer the visual elements of a set are to one another, the more strongly we tend to group them perceptually."[38] This principle is at work when we group the nine *X*s below into sets of three, two, and four.

XXX XX XXXX

In musical pitch space, something akin to the proximity principle is at work when we group pitches that are close in register. However, music also invokes the harmonicity principle: that we group pitches that belong to the same overtone series. Harmonicity could be perhaps be seen as a special case of the Gestalt principle of similarity, which states that we group objects with similar features—for example, when we group together the white and black dots below into separate rows. Because of a shared trait, their color, the black dots are grouped together rather than with the white dots, even though they are not closer in space.

○○○○○○○○○
●●●●●●●●●
○○○○○○○○○
●●●●●●●●●

Through harmonicity, the overtone series suggests a close relationship between pitches (analogous to visual similarity) that is not merely due to proximity. As Andrew Mead has pointed out, the difference between these two types of "closeness" is essential to the organization of tonal music. Vocally, it is easiest to move from one note to another nearby note—this stepwise motion reflects the "proximity" principle. However, we can

[38] Bregman, op. cit., 18-19. Both Bregman and Tenney are influenced by Gestalt psychology—Tenney's *META +HODOS* (Lebanon, New Hampshire: Frog Peak Music, 1986) is an early attempt to translate Gestalt principles into music theory.

more easily relate notes *harmonically* when they are separated by larger intervals (skips) like the third, fifth, or octave. As Mead writes,

> the difference is suggestive with regard to musical syntax: *the most closely related notes are not those that are easiest to get to.* We are fortunate for this difference. Indeed, one might argue that tonality depends on this difference, that Schenker's "Chord of Nature" is very much of *our* nature, our physiology, and that the simplest motions away from it (voices moving by step) reflect what is physically the easiest thing for voices to do.[39]

Through the principle of harmonicity, we select the just intervals as points of reference in the continuum of pitch space. These harmonic relationships suggest a wealth of analytical possibilities that are overlooked by a distance-only abstraction of pitch connections.[40]

Fedor Tau: "Steppe Kargiraa"

The close relationship between the overtones of a musical sound and the familiar (just) intervals shared by many musical cultures is made viscerally audible in the Tuvan vocal style known as *xöömei* (also spelled *höömii* or *khoomei* or referred to as "throat singing"), one of the most remarkable melodic exploitations of the overtone series of a complex sound.[41] A *xöömei* singer sustains a long, raspy drone, shaping his mouth and throat to emphasize individual partials that are already present in the vocal sound. By

[39] Mead, op. cit., 8.

[40] A parallel instance of simple ratios as the preferred referential points in a real number continuum is the representation of rhythmic durations in Western notation as simple fractions of a basic beat. Thus, we often transcribe even eighth or sixteenth notes for a rhythm that in real performance does not divide the beat so precisely (and of course no human performance produces precisely equal divisions). The biases of the notational system seem to suggest that at some level we mentally represent rhythms as simple durational ratios—even when this does not reflect the complex timing of the actual sounding durations.

[41] See Theodore Levin with Valentina Süzükei. *Where Rivers and Mountains Sing: Sound, Music, and Nomadism in Tuva and Beyond* (Bloomington and Indianapolis: Indiana University Press, 2006). A related form of vocal performance is practiced in Tibetan Buddhist chant.

adjusting the vocal cavity, the singer creates different *formants* (peaks in the resonant response of a physical system which amplify vibrations in a certain frequency range): a harmonic whose frequency is near the peak of a formant will stand out from its neighbors. In *xöömei* singing, these formants make a single partial much louder than those surrounding it—the exaggerated difference in amplitude between partials keeps the sound from fusing into a single complex pitch, as is usual for vocal sounds. Rather, the emphasized partial stands out as a independent, whistle-like tone.[42]

Because the melody of a *xöömei* performance is built of the partials of a single fundamental, we can expect the intervals between pitches to be expressible as simple just ratios. In "Steppe Kargiraa," recorded by the singer Fedor Tau, the melody is built on the 8^{th}, 9^{th}, 10^{th}, and 12^{th} partials of a drone of about 66 Hz (C_2+16¢): these partials are equivalent to the first, second, third, and fifth degrees of a major scale in just intonation. Figure 1.6 shows a sonogram of the three vocal phrases in this performance; numbers on the left side label the partials.[43]

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[42] The acoustics of *xöömei* are discussed in William Hartmann's *Signals, Sound, and Sensation* (Woodbury, New York: American Institute of Physics, 1997): 124-25.

[43] The recording analyzed here is commercially released on the CD *Tuva: Voices from the Center of Asia,* produced by Eduard Alexeev, Zoya Kirgiz, and Ted Levin (Smithsonian/Folkways Recordings SF40017, 1990).

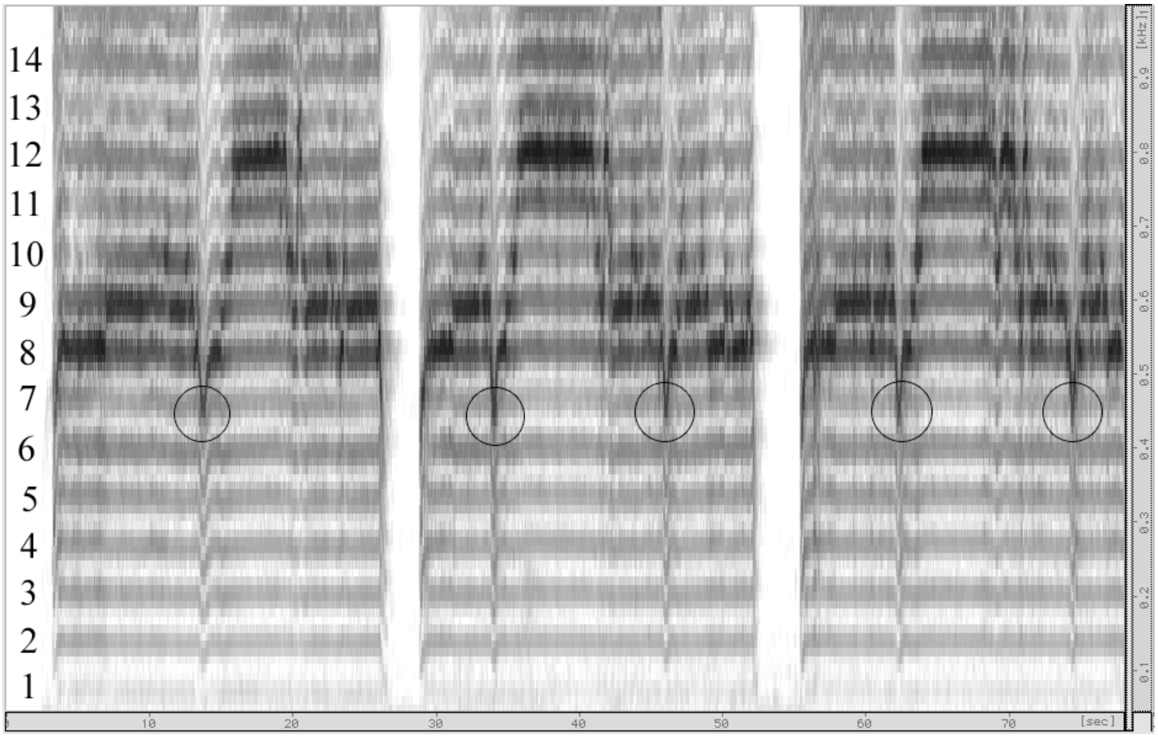

Figure 1.6: Sonogram of "Steppe Kargiraa"

In the sonogram, the darker shades represent greater intensity—these are what we hear as the melody. The fundamental (partial number 1) is relatively weak. The focus on the 8^{th}, 9^{th}, 10^{th}, and 12^{th} partials avoids the less familiar (to Western ears) 7^{th} and 11^{th} partials. There is a distinct perceptual bias toward hearing intervals which are composed of simple factors as the most stable. In many cases, the particular octave the partial appears in seems less important than the partial's pitch class in determining its stability—we can coin the term "partial class," by analogy to pitch class, to refer to a partial under octave equivalence. In terms of the partial number, this will mean the lowest possible representation of the pitch as well as all the multiples of that number by powers of 2: thus, the partial class of the fundamental includes the partials 1, 2, 4, 8, 16, etc., while the partial class of the upper fifth would include 3, 6, 12, 24, etc. The idea that all pitches of a given partial class share certain harmonic qualities is confirmed in our reading of *Steppe Kargiraa*—here, the 8^{th} partial is treated as a "tonic" for the melody, recurring at the beginning and end of each phrase. We tend to experience the 8th partial as more stable than the seventh, even though the 7th occurs lower on the overtone series; this is because the 8th partial shares a partial class with the fundamental, 1, and inherits the fundamental's stable, "rooted" quality.

A few times in each phrase, the singer's voice drops—on the sonogram, this appears as a dip in all the partials at once (the dips are circled in the sonogram). It's interesting to note that even though the seventh partial isn't used in the melody, that this dip very closely approximates an 8/7 interval; this is visible in the sonogram as the eighth partial of the voice falls to the pitch formerly occupied by the seventh. While an in-depth investigation of the pitch language of *xöömei* is beyond the scope of the present discussion, one can speculate that attunement to the partials of the overtone series could well lead a *xöömei* singer to an intuitive use of the seventh partial as a natural adjunct to the pitches of the overtone melody. (Levin notes the use of the pitch equivalent to the seventh harmonic in the Tuvan song "The Orphan's Lament," which is not performed as *xöömei.*) The extension of pitch materials to include intervals using the seventh partial (and beyond) is the subject of the second basic premise of this chapter.

Premise 2: The principles of standard just intonation can be extended to include higher prime numbers.

In the recording of *Steppe Kargiraa* discussed above, the performer builds a melody by selectively emphasizing pitches in the overtone series of a sung drone—the performer, Fedor Tau, limits his melodic pitch material to the partials 8, 9, 10, and 12. When the positive integers—whether understood as partial numbers or components of frequency ratios—are taken as the basis of an interval system, the possibilities for pitch combination are infinite, just like the set of positive integers. Throughout history, composers and theorists have found it necessary to impose limits on this proliferation of possibilities. Sometimes, the explanations offered for such limits seem arbitrary—for example, Johannes Kepler matched the consonant ratios of Western music to the geometrical shapes which can be inscribed in a circle to "prove" that all other ratios were dissonant—but the underlying need to control the number of acceptable intervallic ratios is a serious and persistent problem. In this section, I make the case that the properties of sensory consonance, tunability, and stability that we prize in traditional just intonation (with ratios containing only multiples of 2, 3, and 5) are preserved when ratios include multiples of higher prime numbers: 7, 11, 13, and beyond. The extension into higher primes is not infinite—instead, the comprehensibility of extended just intervals declines in proportion to the complexity of the intervals' ratios. Still, as we shall hear in two musical examples, a modest extension of just intonation into higher primes can offer new and aurally convincing referential intervals.

György Ligeti: Viola Sonata (1991-94), I. Hora Lungă

The first movement of Ligeti's Sonata for Solo Viola, "Hora Lungă," draws on a similar pitch language as the melody of "Steppe Kargiraa"—the basic pitch set of the piece is made up of pitches in the overtone series of F (though pitches from outside the overtone series are occasionally added). Figure 1.7 shows the pitches of the first six measures of "Hora Lungă."[44]

[44] Note that this collection is similar to the equally-tempered "acoustic scale," which analysts have located in music by composers ranging from Bartok to Messiaen. Bob Gilmore discusses this movement of the Viola Sonata in his article "The Climate Since Harry Partch," *Contemporary Music Review* 22/1-2 (2003): 15-23.

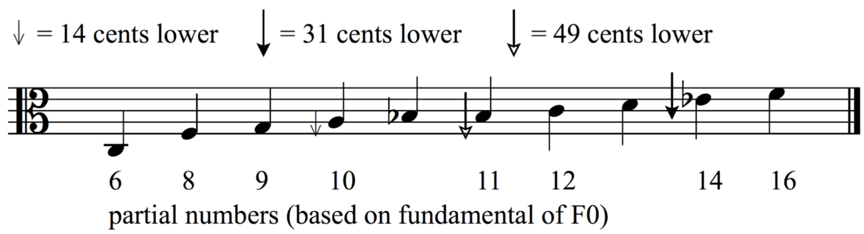

Figure 1.7: chart of pitches in “Hora Lungă,” mm. 1-6

We can imagine an overtone singer performing the beginning of the viola’s melody, building the melody from the overtones of a drone two octaves and a perfect fifth below the viola’s open C string. There are a few pitches in the first six measures, though, that the overtone singer will find that he can’t produce—the D-natural and B-flat, which aren’t overtones of F.

The D-natural (notated without any microtonal inflection), is too high to be the thirteenth partial, which is what one might expect given the context, a scalar rise through partials 8, 9, 10, 11, and 12. The thirteenth partial is 841 cents above the fundamental, while the unaltered D natural is 900 cents above. The 900 cent major sixth in this context seems more likely to be heard as a tone representation of the 27th partial (four octaves and 906 cents above the fundamental). This might seem like a leap to a distant part of the overtone series—but in fact, when understood as the 3rd partial of the 3rd partial of the 3rd partial (3×3×3), we can see that the 27th partial is only a few small steps in harmonic space from the fundamental. (See the discussion of different ways of measuring distance in Chapter 2—the question is often how to compare the effect of a few simple steps to that of one complex step.) Another interpretation of the D is as a just 5/3 sixth (884 cents) above F, though this interpretation means that the lower note of the sixth, F, is no longer viewed as a fundamental at all, but as the third partial of an implied, but unheard fundamental B-flat an octave and a fifth below.[45] The number of plausible tone representations for the tempered major sixth—as the interval 13/8, 27/16, or 5/3, indicates the importance of context in determining how we comprehend interval meaning. In the Ligeti sonata, the most important contextual clue to the harmonic interpretation seems to be the whole step to the D from the preceding C—if we hear that whole step as a 9/8 major whole step (echoing the F-G whole step a fifth below), the 27th partial interpretation is strongest.

In measures 3 and 4, an uninflected B-flat temporarily takes the place of the B-minus-49-cents of the opening pitch collection. Here, the B-flat is unambiguously a perfect fourth above F—but its origin cannot be found in the overtone series. Ligeti replaces a pitch that *is* an overtone of F for one that *has* F as its third overtone (allowing octave equivalence in both cases). This distinction between being and having (*sein* and *haben*) is essential to Hauptmann's nineteenth-century dualist theory of harmony: he used this distinction to build the minor chord as a mirror image of the major. Like the diatonic major scale, Ligeti's scale including the B-flat and D combines overtones of F and tones *of which* F is an overtone, without compromising the overall sense of F as tonic. (This type of relationship is effectively modeled by "lattices" representing just relationships, such as the Riemannian *Tonnetz* discussed in the following section.)

* * *

When Western music theorists have tried to justify musical practice by reference to the natural overtone series, the question of where to stop has often been a vexing problem. If, for example, a theorist claims that the partials one to six (or the string length

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45 Since all just intervals can be found in an overtone series, each just interval implies a fundamental pitch equivalent to the fundamental of that series. This is discussed in more detail below on pages 51-56.

ratios involving those numbers) are the basis of harmony and scales, why not seven or eleven or seventeen? The cutoff can seem arbitrary—for example, contrary to all scientific evidence, Rameau claimed that the *corps sonore* vibrated to produce overtones only at the third and fifth partials, not at any other frequencies.[46]

As discussed in the introduction of this chapter, the legacy of Greek theorists in the Middle Ages was the Pythagorean matching of number ratios with music: the modes of musical practice could be constructed by the cyclical application of the interval 3/2 (the perfect fifth). This produced a *Pythagorean* tuning, in which all intervals could be described as ratios built of multiples of 2 and 3: see Figure 1.8, which shows both the ratio intervals between adjacent pitches and cent values in relation to C. Using the terminology introduced by composer Harry Partch in his book *Genesis of a Music,* we can refer to this as a 3-limit tuning (the highest prime number factor of all ratios is 3).[47] The Pythagorean scale includes an 81/64 ditone (408 cents, a little larger than a tempered major third) and the semitone 256/243, the difference between the ditone and a perfect fourth.

C	9/8	D	9/8	E	256/243	F	9/8	G	9/8	A	9/8	B	256/243	C
0		*204*		*408*		*498*		*702*		*906*		*1110*		*1200*

Figure 1.8: Pythagorean tuning

Despite the counterarguments of Aristoxenus (whose work would remain highly relevant, especially for practical musicians), Pythagoras's equation of musical intervals with the ratios built up of the numbers 1 to 4 became widely accepted as the theoretical underpinning of Western intonation, and dominated the theoretical landscape until the

46 Christensen, op. cit., 155ff.

47 Harry Partch, *Genesis of a Music: An Account of a Creative Work, Its Roots, and Its Fulfillments,* second edition (New York: Da Capo Press, 1974).

fifteenth century. By then, this inherited music theory had come into serious conflict with compositional and performance practice. The problem was this: musicians had begun to treat the interval of the major third as a consonance, but in Pythagorean notation the third was represented by the thorny ratio 81/64 (the simplicity of an interval's ratio is directly correlated with the interval's smoothness to the ear). As early as the twelfth century, the British theorist Theinred of Dover (later seconded by Walter Odington) suggested that the simpler ratio 5/4 might be closer to that used in practice.[48] The difference between the two intervals is easy to hear: the Pythagorean third is considerably higher than any of the thirds used in modern tuning (at 408 cents), and much higher than the mellow, smooth 5/4 just third (at about 386 cents).[49] The idea of allowing the number 5 to participate in defining interval was pushed into broader circulation though the work of Bartolomeus Ramis de Pareia, whose treatise *Musica Practica* (1482) set off a contentious debate with Pythagorean tuning advocates like Gaffurius.[50] Figure 1.9 illustrates a just major scale on C: the numbers between the note names show Ramis's monochord measurements, the second row shows tunings in cents.

C	9/8	D	10/9	E	16/15	F	9/8	G	10/9	A	9/8	B	16/15	C
0		*204*		*386*		*498*		*702*		*884*		*1088*		*1200*

Figure 1.9: Just intonation major scale

Figure 1.10 offers a simpler way to visualize the tunings of this scale. The Pythagorean scale is a collection of seven pitches related by fifths: each fifth in the diagram is a step to the right. In the just intonation scale, the shaded pitches (A, E, and B)

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[48] Mark Lindley, "Just Intonation," *Grove Music Online,* http://www.grovemusic.com, accessed April 15, 2008.

[49] The 22 cent difference between the two thirds (81/80) is called the syntonic comma.

[50] Lindley, op. cit.

are part of a different chain of fifths, related by just 5/4 third to the original chain. Thus, A is a just third above F, E is a just third above C, and B is a just third above G. Note that by allowing the higher prime limit of 5, a much simpler interval can represent the major third: 5/4 instead of 81/64. A parallel could be seen with the Ligeti example: using only factors of 2, 3, and 5, the interval of 969 cents can only be described by the very complex ratio representation 225/128 (C-A-sharp), but if we accept an extended just intonation system including multiples of 7, the interval 969 can be understood as the much simpler ratio 7/4. What was complex becomes simple, though at the cost of allowing higher primes: this theme will recur throughout this study (see especially the discussion of the second preference rule in Chapter 2).

Pythagorean:

	F	**C**	**G**	**D**	**A**	**E**	**B**

Just:

A	**E**	**B**	
F	**C**	**G**	**D**

Figure 1.10: Comparison of Pythagorean and just diatonic scales

Introducing a new axis of thirds and allowing both axes to extend indefinitely results in a figure called the *Tonnetz* (see Figure 1.11); this illustration of just-intonation relationships based on 3 and 5 is best known in its application by nineteenth-century German theorists like Oettingen and Riemann.[51] The *Tonnetz* arranges the just intonation scale of Ramis de Pareia into a two-dimensional grid, where horizontal steps represent 3/2 perfect fifths, and vertical steps represent just 5/4 major thirds. Numbers below each note name show its pitch class in cents relative to C = 0.

[51] The historical origins of the *Tonnetz* are explored in Richard Cohn's "Introduction to Neo-Riemannian Theory: A Survey and a Historical Perspective," *Journal of Music Theory* 42/2 (1997): 167-180. "Neo-Riemannian" theory, the mathematically formalized adaptation of *Tonnetze* to equal-temperament rather than just intonation, has been a much-discussed subfield of theoretical research in the last twenty years. See also Kevin Mooney, "The 'Table of Relations' and Music Psychology in Hugo Riemann's Harmonic Theory." Ph.D. diss., Columbia University, 1996.

F#	C#	G#	D#	A#	E#	B#	Fx
568	*70*	*772*	*274*	*976*	*478*	*1180*	*682*
D	**A**	**E**	**B**	F#	C#	G#	D#
182	*884*	*386*	*1088*	*590*	*92*	*794*	*296*
B♭	**F**	**C**	**G**	**D**	A	E	B
996	*498*	*0*	*702*	*204*	*906*	*408*	*1110*
G♭	D♭	A♭	E♭	B♭	F	C	G
610	*112*	*814*	*316*	*1018*	*520*	*22*	*724*

Figure 1.11: Just intonation *Tonnetz*

Note that the just major scale occupies a small region of a potentially infinite fabric of just intervals. Recent "neo-Riemannian" theorists have explored a tempered version of the *Tonnetz,* in which each dimension returns to its starting point, curling the plane of the *Tonnetz* into a doughnut-shaped torus. In the small excerpt of the just intonation *Tonnetz* illustrated above, there are several different versions of some pitch names: for example A (906 cents) and A (884 cents). These pitches are separated by the syntonic comma 81/80, an interval of approximately 21.5 cents. The syntonic comma is the difference between the 906 cent A (four perfect fifths above F) and the 884 cent A (a just major third above F). (The need for two versions of certain pitches to assure just tuning in simple tonal progressions led to innovative keyboard designs such as the sixteenth-century composer and theorist Nicola Vicentino's archicembalo with thirty-six keys to the octave.)

La Monte Young: *The Well-Tuned Piano* (1964-)

La Monte Young's epic piano solo *The Well-Tuned Piano* extends the emphasis on the "out-of-tune" upper partials heard in Ligeti's Viola Sonata to a complex tuning system based entirely on the intervals 7/4 (the flat "septimal" minor seventh) and 3/2 (the perfect fifth).[52] The melodic intervals of the sonata are abstracted to build a pitch lattice whose geography Young explores over the course of a carefully-planned five-hour improvisation. The title's playful reference to Bach's *Well-Tempered Keyboard* draws attention to an important distinction: Young's tuning is a just tuning, without the compromises inherent to any tempered system.

The *Well-Tuned Piano* tuning takes as its theoretical departure point the *Tonnetz* of Figure 1.11. Young professes a dislike for the intervals based on the integer five (Gann 135); in his tuning lattice for *The Well-Tuned Piano*, he replaces the 5/4 major third with the 7/4 septimal minor seventh.[53] Just as the just diatonic and chromatic scales can be understood as regions within an infinite harmonic fabric, Young's twelve-note scale is a subset of the harmonic space defined by 7 and 3. The replacement of 5 by 7 gives the tuning an exotic quality, as the unusual intervals whose ratios include the integer 7 (7/4, 7/6, 9/7, etc.) mingle with the clearly ringing perfect fifths—which are subtly different from the slightly smaller (by two cents) fifths of standard twelve-tone equal temperament. Figure 1.12 diagrams Young's tuning—the numbers beneath the notes indicate the pitch class in cents (based on A 440 Hz = 900 cents). The up and down arrows show a deviation from the equal-tempered version of the pitch class by approximately 31 cents (the difference between an equal-tempered and septimal minor seventh)—thus, the interval D to C⇓ is 1000 cents (D to C) minus 31 cents, or 969 cents.

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[52] My comments on *The Well Tuned Piano* in this section draw extensively on Kyle Gann's article, "La Monte Young's *The Well-Tuned Piano," Perspectives of New Music* 31/1 (Winter, 1993), 134-162. Gann was the first musicologist to reconstruct the details of Young's tuning, which the composer had kept secret since the 1960s.

[53] By using 7/4 *instead* of 5/4 rather than in addition to it, Young avoids the need to move to a three-dimensional *Tonnetz.* For more on multidimensional *Tonnetze,* see my discussion of harmonic space models devised by James Tenney and Ben Johnston in Chapter 3, and also Edward Gollin, "Some Aspects of Three-Dimensional *Tonnetze," Journal of Music Theory* 42/2 (1998): 195-206.

The relationship of all the pitches by intervals of 3/2 or 7/4 means that each pitch can be understood as an overtone, however distant, of the pitch D⇑; numbers in parentheses show the partial number of each pitch, based on D⇑=1.[54]

B♭⇓	F⇓	C⇓	G⇓	
963	465	1167	669	
(49)	*(147)*	*(441)*	*(1323)*	
C	G	D	A	E
1194	696	198	900	402
(7)	*(21)*	*(63)*	*(189)*	*(567)*
D⇑	A⇑	E⇑		
225	*927*	*429*		
(1)	*(3)*	*(9)*		

Figure 1.12: La Monte Young's *Well-Tuned Piano* pitch lattice

D⇑	E	E⇑	F⇓	G⇓	G	A	A⇑	B♭⇓	C⇓	C	D
225	402	429	465	669	696	900	927	963	1167	1194	198

Figure 1.13: *Well-Tuned Piano* "chromatic scale"

Figure 1.13 arranges the pitches in scalar order (the cents values refer to pitch classes, which is why the final D has a lower number than the C which precedes it). Compared to the nearly even intervals of the 5-limit chromatic scale, consecutive intervals in this "chromatic" scale vary greatly in size, ranging from 27 cents to 204 cents (Gann, 137). In practice, each pitch class is mapped onto one of the keys of the piano—Young devises this mapping so that pitches on each axis of fifths are related by fifths on the keyboard (Figure 1.14). The result, though, is that intervals on the vertical (7/4) axis are played on the keyboard as spans of either a major sixth or minor seventh. To sidestep this inconsistency, I will refer to all pitches by their actual sound, as in Figure 1.12 and 1.13, not by their keyboard mapping (as in the transcriptions of Gann's article).

[54] Though Young's approach (like all just intonation) can be understood as expressing an overtone-series structure, his reliance on octave equivalence separates his theory from any specifically "spectral" tendencies. Young is less interested in replicating the overtones of natural sounds than creating a grid of audible just interval relationships.

B	F#	C#	G#	
963	465	1167	669	
C	G	D	A	E
1194	696	198	900	402
E♭	B♭	F		
225	927	429		

Figure 1.14: Keyboard mapping (after Gann)

Young's tuning is replicated in all the registers of the piano—as a result, the overtone-based relationships are freed from their usual connection to register. For example, the interval from D⇑ to F⇓ (1 to 147) need not be arranged with the D⇑ as a fundamental, and the F⇓ as the 147th partial seven octaves and 240 cents above. Rather, the pitches can be in the same octave—a 147/128 interval of just 240 cents—or even inverted with the F⇓ lower than the D⇑, as a 960 cent interval, 256/147.

How do we hear such complex intervals? Is this relationship simply too complex to comprehend as a just interval in the same way that we immediately apprehend ratios like 4/3 or even 11/8? And aren't we likely to hear the interval of 240 or 960 cents as representing a far simpler integer ratio, slightly out of tune—8/7 (231 cents) or 7/4 (969 cents)? These questions are best answered, I think, by reference to the context in which the interval occurs—context is of utmost importance in determining our tone representation of a given interval. If we hear the interval F⇓-D⇑ with no other sonic information, then we're most likely to understand it as representing the 7/4 septimal minor seventh—or even an out-of-tune minor seventh from a more standard five-limit just tuning (16/9, or 996 cents) for example, even further out of tune. But the simple intervallic building blocks in Young's tuning can combine to lead us securely into very distant harmonic territory—we have to negotiate two 7/4 intervals and one 3/2 to reach the F⇓ from the D⇑; but if these intervals are introduced carefully, it may be possible to follow the path on the harmonic lattice from 1 up to 147 through the intermediate points. The power of contextual relations makes questionable any absolute line limiting the perceptibility of complex intervals.

Young rarely demands that we immediately relate such distant points in the tuning—and when he does, he often connects them in such a way that the intermediate points from one to the other are obvious. For an example of his way of using these intervals musically, let's turn to two excerpts from the opening of the piece, during which Young makes a very gradual transition from his "Opening Chord" to the "Magic Chord" (these chords and their labels are identified in Gann's article).[55]

The opening chord is the subject of extended improvisation for almost ten minutes—the transcription in Figure 1.15 shows the first two minutes. This transcription includes the beginning of what Young calls the "Theme of the Dawn of Eternal Time"—all of the major sections and themes of the piece receive titles, ranging from the mundane "Opening Chord" to the poetic "The Goddess of the Caverns Under the Pools." This section strongly projects the tonal center D⇑, which (as noted above) is the theoretical fundamental for the entire tuning system. Its root status is strongly supported by the A⇑ a fifth above and the tonic-dominant effect of the alternating D⇑ and A⇑ in the lower voice. (The idea of rootedness will be explored in more detail in the next section.) In this context, C is associated strongly with D⇑ as a natural seventh—an apparently consonant

[55] As the work has no fixed score, the analytical notes which follow are based on the five CD Gramavision recording of a performance on October 25, 1981 (Gramavision 18-8701-1). This is the same recording addressed by Gann in his groundbreaking article. A DVD video recording, which combines Young's performance with a light installation by Marion Zazeela, was released in 2002 by Young's MELA Foundation: the complete title is *The Well-Tuned Piano in The Magenta Lights (87 V 10 6:43:00 PM –87 V 11 01:07:45 AM NYC)* (Catalog number JD 002).

tone that does not call for a resolution. Large 9/7 major thirds and 8/7 "whole steps" appear melodically.

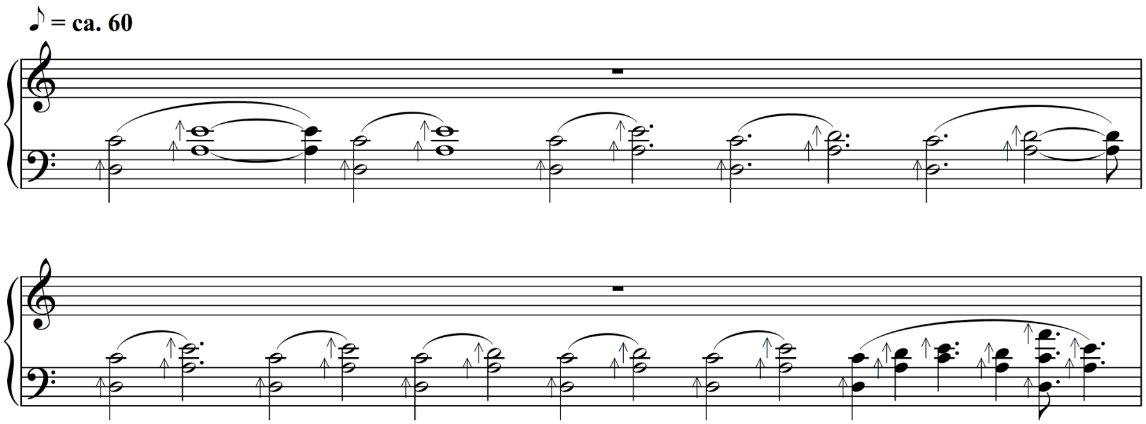

Figure 1.15: Opening of *The Well-Tuned Piano*, 0'00"-1'50"

The small area of tonal pitch space described by the four pitch classes of the opening chord is expanded by the addition of B-flat⇓ and D at 9'38": see Figure 1.16 for a transcription. Figure 1.17 illustrates both the opening set of four pitches and the expanded set of six in their lattice configuration. After more than nine minutes devoted to the opening chord, the introduction of the 9/7 third B-flat⇓-D is startling in its novelty. As we listen on, ways that these new pitches might fit into the harmony begin to become apparent. In size, this third matches the C-E⇑ third that has appeared frequently—the new third, however, is separated by an unusual interval of 231 cents (8/7) from C-E⇑. This reflects a move of one vertical step on the pitch lattice—a move along the 7 axis.

Figure 1.16: Introduction of new pitches B♭⇓ and D, 9'15"-9'58"

The B-flat⇓ is removed from D⇑ by two steps on the 7 axis—thus, in one sense we can only reach the B-flat⇓ "through" C. Another way of putting this is to observe that the integer 49 which represents B-flat⇓ has only one factorization: 7 x 7. (A comparable case on the 5/3 lattice is Rameau's enharmonic quintuple proportion 1:5:25 or C-E-G-sharp; the G-sharp can be related to C only through the intermediate E.) The two new pitches, however, come very close to simpler just intervals—it is conceivable that we could hear the B-flat⇓ as an out-of-tune unison with A⇑ or an out-of-tune fifth above D⇑ (in both cases, 49/48 or 36 cents sharp), or the D as a near unison/octave with D⇑ (separated by only 64/63 or 27 cents). Whether the new pitches are heard as representing the complex ratios of 49/32 and 63/32 in relation to the fundamental D⇑ depends largely on musical context: in this case, Young offers enough parallels between the new pitches and those we've already heard to allow the ear to make these esoteric connections. The complex intervals are convincingly broken down into simpler steps of 3/2 and 7/4.

C *1194* *(7)*		
D⇑ *225* *(1)*	*A⇑* *927* *(3)*	*E⇑* *429* *(9)*

B♭⇓ **963** ***(49)***		
C *1194* *(7)*		**D** **198** ***(63)***
D⇑ *225* *(1)*	*A⇑* *927* *(3)*	*E⇑* *429* *(9)*

Figure 1.17: Lattice representations of pitch collections in Figures 1.15 and 1.16

A closely-related tuning has been designed by Young's student Michael Harrison, a fellow disciple of Pandit Pran Nath and the only pianist other than Young to perform the *Well-Tuned Piano*. Like Young, Harrison shuns the thirds and sixths based on 5, and uses only perfect fifths and natural sevenths in his tuning matrix (see Figure 1.18).

E♭⇓	B♭⇓	F⇓	C⇓	G⇓		
267	969	471	1173	675		
F	C	G	D	A	E	B
498	0	702	204	906	408	1110

Figure 1.18: Michael Harrison's "Revelation" tuning

Harrison is particularly interested in the smallest intervals available in this tuning—what he terms the "celestial commas" between F⇓—F, C⇓—C, and G⇓—G. When sounded simultaneously, these intervals of 64/63 (27 cents) produce complex timbres and compelling beating patterns: Harrison is interested as much in the pure sonic quality of these just intervals as their syntactical meaning as rationally derived pitches in an extended just intonation scale.[56]

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[56] Harrison describes this tuning in a program note for his solo piano work *Revelation* (see http://www.michaelharrison.com/revelation-program-notes.html (accessed April 15, 2008). His interest in the timbral effects of commas aligns Harrison with the type of microtonality Georg Friedrich Haas defines

Premise 3: Each just interval implies a fundamental or root, and a specific closely-related harmonic domain based on the overtones of that fundamental.

In the previous sections, we've explored how just intervals (whether within or beyond the traditional five-limit) can serve as referential intervals in a variety of styles, and offer such musically desirable qualities as smoothness and stability. An additional property of just intervals is their tendency to imply a *root*—this is essential to common practice tonal music, and is a property of ratio intervals which is lacking in the atonal world where interval is conceived purely as distance.

Rootedness is the basis of root our musical intuition that when we hear the pitches C and E, they not only "belong together" (as an instance of the just major third), but also share a common root of C. To the concept of rootedness we can add the idea of harmonic relatedness: that intervals sharing a root implication can be effectively combined with one another: for example, because like C-E, C-G implies a root of C, it can be combined with C-E to create a triad with similar harmonic meaning to either interval on its own. An effect of rootedness is that we *group* pitches together which can fit into the same overtone series: this suggests a mechanism of "harmonic templates" supported by the research of Albert Bregman and Ernst Terhardt. Pitches fitting into the same harmonic template will be grouped together in our aural perceptions.

In our discussion of *Koan,* we saw how the overtone series of a single complex sound had deep implications for the consonance or dissonance of pairs of complex sounds. In the following examples—Lucier's *Music on a Long Thin Wire* and Tenney's string quartet version of *Koan*—we will see some ways that composers have exploited the

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as *Klangspaltung* or tone-splitting: see Haas, "Mikrotonalitäten," in *Musik der anderen Tradition: Mikrotonale Tonwelten* (Munich: Musik-Konzepte, Edition Text+Kritik, 2003), 59-65.

overtone series, and our perceptual bias for grouping pitches related by simple integer ratios. Our discussion will begin with the simplest, most "natural" use of the overtone series, Lucier's *Music on a Long Thin Wire,* and progress to the more abstract treatment of the overtone series in *Koan for String Quartet.*

Alvin Lucier: *Music on a Long Thin Wire* (1977)

Though overtones play a central role in the way we understand consonance between complex tones, listeners usually aren't aware of the individual overtones of a sound. Overtones combine to create the perceived overall timbre of a sound, but are not recognized as separate entities. Alvin Lucier's 1977 sound installation *Music on a Long Thin Wire* plays on the ambiguous nature of overtones—are they independent "voices" or merely components of a larger whole? The installation makes the upper overtones of a vibrating string clearly and distinctly audible, even as we perceive them fitting into an overall timbral "harmony."

Lucier has "performed" the piece in many different versions, but the basic premise remains the same. A electrically-conductive wire up to thirty meters long is stretched over two bridges, with its ends attached to the electrical outputs of a sine-wave oscillator. A magnet is placed at one end of the wire so that, as the oscillator varies the current in the wire, electromagnetic induction causes the wire to vibrate. The sound of the vibrating wire is picked up by microphones and amplified through loudspeakers.

This setup acts as an aural microscope, magnifying the partials of the vibrating string so that they can be clearly heard. The long wire is sensitive to tiny variations in the space and to subtle changes in the electromagnetic field—as a result, the string drifts from one mode of vibration to another, creating changes in the volume of the partials of the sound. In his original conception of the piece, Lucier played the wire as an instrument, changing the frequency and amplitude of the sine-wave oscillator to build musical "phrases" in the wire's response. In subsequent performances, though, he found such obvious manipulations too predictable and artificial. Instead, he set up the piece with a given oscillator frequency, then simply allowed the string to react on its own—this aesthetic is summed up by the title of his essay, "Careful Listening is More Important than Making Things Happen."

> Now, I made a recording of *Music on a Long Thin Wire* and had set it up in a beautiful space in New York, the U. S. Custom House, that has a huge dome on the top floor. I had stretched the wire about thirty meters, which is about as long as I've ever stretched it, and had decided that I wasn't going to perform it. I would be true to my ideal. I wouldn't change anything once I had tuned it and set the volume level. I would simply see what happened. I promised myself that for a couple of reasons. One, is that you want to be true to your idea, otherwise there's no integrity to your work. The other is that I knew it wouldn't sound right. If you change something in the middle of a recording, it's usually a mistake.[57]

* * *

Listening to Lucier's recording of *Music on a Long Thin Wire* offers new insights into the way we hear simultaneous pitch combinations. James Tenney's *Koan* showed us how two complex tones in a simple frequency ratio are perceived as consonant—Lucier's piece "zooms in" to focus on the partials of a single complex tone. Because all of the individual overtones we hear in *Music on a Long Thin Wire* are multiples of the same fundamental frequency, the intervals between those overtones are always just, rational intervals—the same just intervals that we hear as consonances when they're played with complex tones. This is the first of many parallels between the sound world *within* a single

[57] Alvin Lucier. "There are all these things happening: notes on installations." In *Reflections: Interviews, Scores, Writings* (Cologne: Musiktexte, 1995): 528-530.

note (that note's overtone structure) and the sound world of many notes in combination (a chord).

When we hear these just intervals between the partials of a complex tone, sensory mechanisms interpret the partials as part of the same entity—without being conscious of it, we group the partials together as emanating from the same source, and extrapolate from their frequencies a likely fundamental frequency for that source. For example, if we hear two simultaneous pure tones at 440 and 550 Hz (a just major third from A to C#), we're likely to hear them as the fourth and fifth partials of a fundamental of 110 Hz (the largest common denominator of the two frequencies). This phenomenon is what makes it possible for the tiny speaker of a telephone earpiece to convincingly reproduce a low bass voice: even though the speaker is too small to convey the voice's fundamental, it can transmit its upper partials—from this data, we mentally recreate the low frequency of the fundamental.[58] Psychoacousticians suggest that we've developed a mental template of the relationships between partials of a complex tone from our frequent encounters with such sounds—and that this template is used to make sense of incoming auditory data.[59] When we find a template that matches the partials that we hear, we perceive the fundamental pitch of the sound. As acoustician William Hartmann explains,

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[58] The telephone can't provide frequencies below about 220 Hz, so whenever we hear a voice on the telephone with a perceived pitch below 220 Hz (the A below middle C), we are using our ability to provide a missing fundamental. This phenomenon can even occur when the harmonics of a sound are presented separately to each ear: see A. J. M. Houtsma and J. L. Goldstein, "The Central Origin of the Pitch of Complex Tones: Evidence from Musical Interval Recognition," *Journal of the Acoustical Society of America* 51 (1972): 520-529. This suggests that the production of the missing fundamental happens not physiologically within either of the ears but rather in the central nervous system. For a more detailed treatment of this topic, see Stanley A. Gelfand, *Hearing: An Introduction to Psychological and Physiological Acoustics,* New York: Marcel Dekker, 4th ed. 2004: 378-79.

[59] Such templates are crucial in separating out simultaneous notes—if we hear two violins playing different notes at the same time, we separate them from one another by subconsciously matching them to different templates.

> Modern theories of pitch perception are foremost pattern matching theories. They assume that the brain has stored a template for the spectrum of a harmonic tone, and that it attempts to fit the template to the neurally resolved harmonics of a tone. Given the ubiquity of periodic tones in the everyday environment, a template for the harmonic spectrum is a reasonable hypothesis. [...] The model can be made quantitative by establishing tolerances for the template for accepting harmonics into an entity.[60]

It's important to note that the harmonics between partials do not need to be precisely in tune to fit into a single complex "entity"—the tolerance that Hartmann mentions will be examined more closely later in this chapter. The effects of this template matching can be clearly demonstrated through an analysis of an excerpt of *Music on a Long Thin Wire.* Figure 1.19 shows a spectrogram of the first 90 seconds of Lucier's U. S. Custom House installation—this excerpt is from the third of four versions of the piece on his 1980 recording.

[60] Hartmann, op. cit., 135.

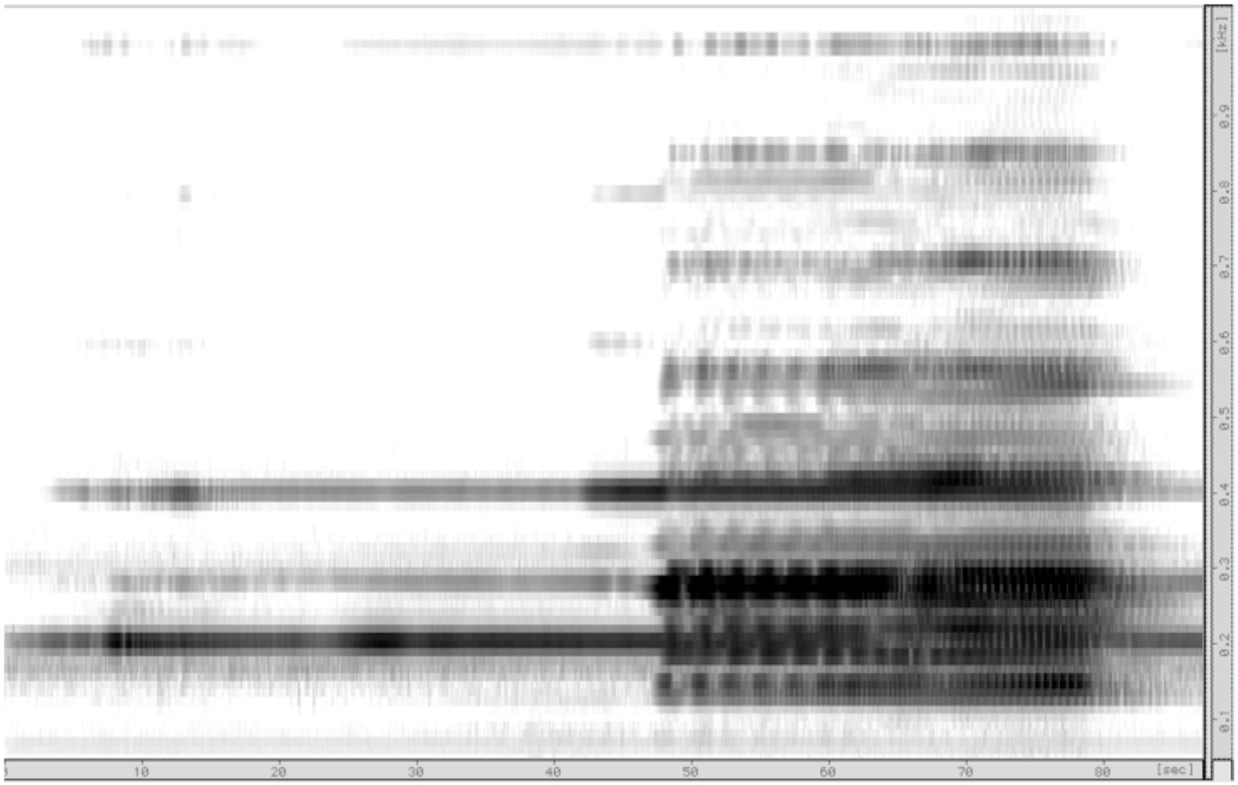

Figure 1.19: Spectrogram of *Music on a Long Thin Wire,* track 3, 0'00" to 1'30"

As the excerpt begins, we hear most prominently partials at approximately 205 and 410 Hz—the upper partial is twice the frequency of the lower, an interval of an octave above. The octave between the pitches strongly suggests a template which takes these frequencies as the first and second partials of a complex tone with a fundamental at 205 Hz—a slightly flat A-flat. However, there's also some sonic evidence against this interpretation: the faint but still perceptible partial at around 274 Hz. This partial doesn't fit into the template based on a 205 Hz fundamental—though it's not loud enough to completely undermine that interpretation. The result, to my ear, is a tense, unstable A-flat—the 274 Hz partial (approximately a fourth above the 205 partial) destabilizes the assumed fundamental without suggesting a new one.

About 45 seconds into the excerpt, the harmony undergoes a sudden change—the harmony is joined by a lower pitch and becomes much richer, while at the same time a series of beats begins to interrupt the sound. (The beats are most likely due to an interference pattern between the resonant frequency of the string and that of the sine wave oscillator driving the vibration.) What's striking here is that the new harmony contrasts with our assumed A-flat fundamental—instead, it seems to be based on the D-flat a fifth below (the effect is not unlike a V to I cadence in tonal music).

As the string shifts into a new mode of vibration, different partials are emphasized, particularly at 137 and 274 Hz. The partials which we'd interpreted as the first and second of an A-flat fundamental—for convenience, I'll notate this hearing as A-flat(1:2)—turn out to be the third and sixth partials of a Db fundamental, part of the chord D-flat(2:3:4:6). This change in interpretation is illustrated schematically in Figure 1.20.

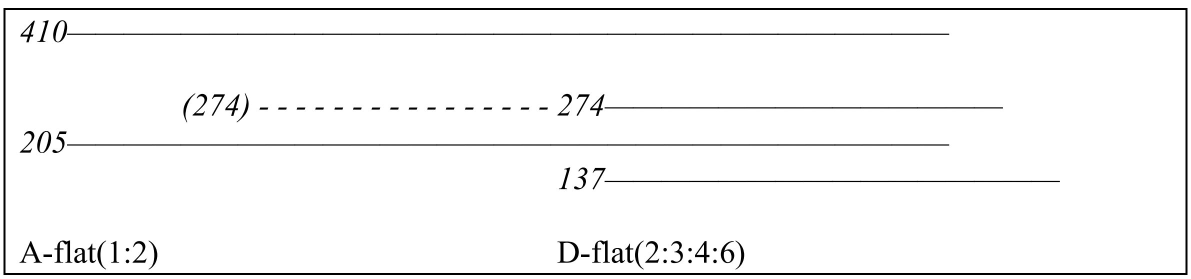

Figure 1.20: Templates and fundamentals in *Music on a Long Thin Wire*

The continuously sounding pitches at 205 and 410 Hz are reinterpreted to fit a new context—as evidence accumulates which is better explained by a new template, our hearing process automatically shifts its perceived fundamental to match the new data.

* * *

The determination of a root in this excerpt from *Music on a Long Thin Wire* depends on a constant comparison of potential harmonic templates. Psychoacoustician Ernst Terhardt has suggested that our strategy of grouping together pitches from the same overtone series is acquired from repeated experience with harmonic sounds, particularly speech (Terhardt, "Pitch, Consonance, and Harmony," 1068). In the mid-1970s, Terhardt developed an algorithm predicting the virtual pitch of a set of partials: virtual pitch is the pitch we perceive for a complex tone (as opposed to spectral pitch, the pitch we hear for any of the pure wave components of the sound). As in our example from *Music on a Long Thin Wire,* Terhardt's algorithm is based on the principle that the equal spacing (in terms of frequencies) of the partials of a harmonic tone allow us to determine the virtual pitch, even when the fundamental itself is not sounding. When extended from sets of partials to actual musical pitches, his theory offers a suggestive model for the way we might find root implications for complex sonorities.[61]

Terhardt's approach is based on two steps: first, the analysis of a complex sound for "determinant spectral components"—the prominent frequency components which contribute to the overall perception of the sound; second, the deduction of virtual pitch or fundamental frequency by subharmonic matching, expressed through an algorithm.[62] Deducing the determinant spectral components is a fairly complex process requiring significant attention to the irregularities and complexities of the physiology of the hearing system—more relevant to our current research is the second part of Terhardt's process. As Terhardt himself points out, it is often sufficient to determine the "nominal virtual

[61] Terhardt himself has explored some of these musical implications in the articles "Pitch, Consonance, and Harmony," *Journal of the Acoustical Society of America* 55 (1974), 1061-1070 and "The Concept of Musical Consonance: A Link between Music and Psychoacoustics," *Music Perception* 1 (1984), 276-295. See also the work of Richard Parncutt, which adapts Terhardt's work for musical applications. Parncutt develops a model which "predicts the number of audible harmonics in complex tones, the multiplicity and tonalness (and hence consonance) of musical tone simultaneities (tones, dyads, and chords), the various possible pitches of simultaneities and the roots of chords." Richard Parncutt, *Harmony: A Psychoacoustical Approach* (Berlin: Springer-Verlag, 1989): 135. See also Richard Parncutt and Hans Strasburger, "Applying Psychoacoustics in Composition: 'Harmonic' Progressions of 'Nonharmonic' Sonorities." *Perspectives of New Music* 32/2 (Summer 1994), 88-129 and "Revision of Terhardt's Psychoacoustical Algorithm of the Root(s) of a Musical Chord." *Music Perception* 6/1 (1988), 65-94. Parncutt's research, particularly the 1988 article, has had a strong influence on Olli Väisälä's research, which is discussed in connection with Premise 4 below.

[62] Ernst Terhardt, "Calculating Virtual Pitch," *Hearing Research* 1 (1979): 155-182.

pitch," ignoring nonlinearities in the hearing system—this nominal virtual pitch is in most circumstances very close to the real one.

The basics of Terhardt's algorithm can be expressed in two principles—these are quoted here in full:

> (a) In first approximation, virtual pitch is specified by any small pitch interval (small region) which comprises at least two subharmonic pitch values of different determinant spectral pitches.
> (b) The precise virtual-pitch magnitude is specified by that subharmonic pitch magnitude within the crucial interval which pertains to the most determinant spectral pitch.[63]

Terhardt's illustration is a complex tone made up of three partials, at 520, 620, and 720 Hz. (520 Hz = C_5-11¢, 620 = E-flat$_5$-6¢, 720 = F_5+53¢). Experimental evidence shows that we hear the virtual pitch of this complex not at 20 Hz—the frequency which is a common factor of each component—but rather at 104 Hz. Terhardt's model also gives this result, based on a comparison of subharmonics. For each determinant spectral pitch 520, 620 and 720, Terhardt determines all of the *subharmonics* of the tone (see Figure 1.21). This is not the same as invoking a dubious "subharmonic series" like nineteenth-century dualist theorists—a better way of understanding this table is that each subharmonic represents a potential fundamental for which the determinant spectral pitch is an upper partial. Thus, 520 can be the second partial of 260, the third of 173.3, the fourth of 130, etc. In its "first approximation," the virtual pitch extends over the region where subharmonics of the determinants fall within a small frequency band—that is, a small band of frequencies which includes fundamentals for each of the determinants understood as upper partials. Here, the range from 102.9 to 104.0 approximates a virtual pitch for which 520 is the fifth partial, 620 is the sixth, and 720 is the seventh. Terhardt

[63] Ibid., 168-169.

notes that in some cases, not all of the determinant pitches will have close subharmonics—the coincidence of just two is sufficient to calculate virtual pitch, though interpretations where more subharmonics coincide are preferred.

subharmonic number			
1	520.0	620.0	720.0
2	260.0	310.0	360.0
3	173.3	206.7	240.0
4	130.0	155.0	180.0
5	**104.0**	124.0	144.0
6	86.7	**103.3**	120.0
7	74.3	88.6	**102.9**
8	65.0	77.5	90.0

Figure 1.21: Table of subharmonics for frequencies 520, 620, and 720

To more specifically determine the virtual pitch, we must select the subharmonic which belongs to the "most determinant" of the three spectral pitches 520, 620, and 720. In general, the lowest of the pitches is taken as most relevant for the determination of virtual pitch: thus here the virtual pitch is 104 Hz, or 520 Hz/4.

In some cases, there will be more than one cluster of subharmonics which might serve as a virtual pitch. In this case, Terhardt offers some criteria by which one is likely to emerge as the perceived fundamental:

1) the most determinant pitch will have a subharmonic that falls in the "integrating interval [the pitch range of "first approximation" described above] comprising the greatest number of near coincidences"—thus we would prefer a virtual pitch which is close to a subharmonic of all three pitches over one which approximates subharmonics of just two.

2) we choose a virtual pitch that corresponds to the smallest subharmonic number—thus, the highest virtual pitch, with the determinant pitches heard as the lowest possible partials.

3) we favor the virtual pitch with the smallest integrating interval—that is, the fundamental where the subharmonics cluster most tightly together.[64]

Note that this approach has a built-in degree of "tolerance"—this is what leads us to accept 104 Hz as the virtual fundamental, even though is not precisely a divisor of 620 or 720, but just an approximation. Criteria 2 and 3 often work against one another—in many cases, one will find a very tight cluster of subharmonics only in the high subharmonic numbers. If we were extremely strict about what constituted "close," accepting only an exact match, we would have to extend the table above to find 20 Hz as a fundamental with 520 as the 26th partial, 620 as the 31st, and 720 as the 36th (Figure 1.22).

subharmonic number			
1	520.0	620.0	720.0
2	260.0	310.0	360.0
3	173.3	206.7	240.0
4	130.0	155.0	180.0
5	**104.0**	124.0	144.0
6	86.7	**103.3**	120.0
7	74.3	88.6	**102.9**
8	65.0	77.5	90.0
...	...	...	...
26	**20.0**	23.8	27.7
27	19.3	23.0	26.7
28	18.6	22.1	25.7
29	17.9	21.4	24.8
30	17.3	20.7	24.0
31	16.8	**20.0**	23.2
32	16.3	19.4	22.5
33	15.8	18.8	21.8
34	15.3	18.2	21.2
35	14.9	17.7	20.6
36	14.4	17.2	**20.0**

Figure 1.22: Expanded table of subharmonics, showing common factor of 20.0

Terhardt typically avoids virtual pitches which demand that the components be heard as extremely high harmonics: as he notes, "there does not exist a sharp boundary toward higher harmonic numbers but rather a decreasing probability of the harmonies to

[64] Ibid., 169.

be, relevant which approaches zero at about the 20th harmonic" (170). Of particular interest, then, is the way that "close" is defined. Terhardt is willing to let this value vary, depending on the application and the type of results desired—his general description is that this "integrating interval" is "the pitch interval in which slightly different pitches perceptually fuse." This can be defined as a percentage of the frequency of the subharmonics—other researchers have suggested that the "mesh size of the hypothetical 'harmonic sieve' for pitch is about ±2%-3% percent" or about a quartertone.[65] For a strongly pitched, harmonic signal, one might demand greater precision, with a smaller tolerance value—for pitches of noisier, less harmonic sounds, a higher degree of tolerance could be permitted. In closing, Terhardt describes how the algorithm can accurately find the root of a major chord from a spectral analysis—this promising analytical possibility will be explored in greater depth in Chapter 2.

Terhardt's research offers support to my premise that in addition to a root, each just interval implies a domain of related pitches—these are exactly those pitches which do not change the root assignment or apparent tonal meaning of any of the existing pitches when added to the input set. This kind of harmonic implication is the basis of James Tenney's *Koan for String Quartet.*

James Tenney: *Koan for String Quartet* (1984)

In most of the examples we've looked at so far, the musical material has focused on the overtones of a single fundamental pitch—whether literally, as in "Steppe Kargiraa" or in a more abstract sense, as in *The Well-Tuned Piano,* where every pitch can be

[65] Brian Moore, Brian Glasberg, and Robert Peters, "Thresholds for Hearing Mistuned Partials as Separate Tones in Harmonic Complexes," *Journal of the Acoustical Society of America* 80 (1986), 479-483.

understood as an overtone (however distant) of the generating D⇑. The exception is Tenney's *Koan,* which, in the course of its glissando through different interval sizes, constantly recasts the violin's two pitches as overtones of different implied fundamentals. We will return to Tenney's *Koan,* this time in his 1984 reworking of the piece for string quartet. Here, the quartet's first violin plays a version of the original 1971 piece, but the other instruments provide a "harmonic meaning" for each of its microtonal intervals, by adding what Tenney describes as "a complex 'chord "progression' on various roots or fundamentals."[66] The implied fundamentals are made explicit by the addition of pitches confirming and elaborating the fundamentals and their overtones.

In the years between the composition of *Koan* for solo violin and *Koan for String Quartet,* Tenney became deeply interested in questions of harmony, and particularly of just intonation. While he had been aware of just intonation theory for many years (beginning probably with his brief spell as an assistant to Harry Partch while studying at the University of Illinois from 1959-1961), it only became a significant part of his own musical language with the orchestra piece *Clang* in 1976. In his 1984 article, "John Cage and the Theory of Harmony," Tenney links Cage's philosophy of music to the just intonation theory pioneered by Partch. The reworking of the solo violin *Koan,* which has strong ties to Cage's aesthetic in its simple score and conceptual purity, as a just intonation string quartet reflects this unlikely pairing.

One of the first changes is the systematization of the microtonal steps of the first violin—while in the solo version, Tenney was content to indicate that each step should be "about an eighth of a semitone," here he specifies that "the intervals played by the first

[66] James Tenney, performance note in score of *Koan for String Quartet.* Analogous shifting fundamentals can be observed in Ligeti's 1971 orchestra piece, *Melodien,* which scholars have tended to approach solely through distance-based analytical methods. An excerpt from *Melodien* is analyzed in detail in Chapter 2.

violin are determined by the simplest frequency ratios within "tolerance" of successive steps of one-sixth of a tempered semitone." Tenney chooses the simplest ratio that approximates each sixth-of-a-semitone step—thus, for example, the first seven ratios are:

3/2	702 cents
40/27	680
22/15	663
16/11	649
13/9	637
10/7	617
17/12	603

Notating the violin's intervals as frequency ratios indicates not only their degree of consonance or stability, but also a specific relationship to a fundamental or root. The other instruments of the quartet reinforce the harmonic implications of the first violin's interval by playing notes from an overtone series based on this fundamental. To retain the "linear and predictable" formal quality of the solo violin work in the quartet adaptation, Tenney uses a few simple rules to generate the pitches of the lower strings. These rules change from section to section, but within each section, their unfolding is as predetermined as that of the soloist. The rules are described in the notes preceding the score. Each of the lower strings plays a pitch equivalent to a theoretical combination tone of the first violin's two pitches. Combination tones are a psychoacoustical phenomenon: under certain conditions two simultaneous tones can generate the sensation of a third (or fourth, etc.) tone which is not actually present in the acoustical signal. The generating tones must be quite powerful to generate the non-linear distortions in the middle ear which create combination tones—they are much less likely to be heard on string instruments than on winds or brass. In this case, Tenney uses the idea of combination tones as a compositional device: sounding combination tones are not produced by the violin's actual sound. We can see his harmonic procedures at work in the following brief excerpt:

Figure 1.23: James Tenney, *Koan for String Quartet,* mm. 183-188. Copyright 1984 by Sonic Art Editions. Used by permission of Smith Publications, 2617 Gwynndale Ave., Baltimore, Maryland.

The first violin begins the excerpt with an 15/11 interval (about 537 cents), a complex ratio which implies a fundamental near the piano's lowest F. The violin's interval gradually narrows to a 4/3 perfect fourth, with an implied fundamental nearly two octaves higher, at the E below middle C. After the perfect fourth, the pitches of the violin continue to contract, reaching 13/10 (454 cents) at the end of the excerpt. The cello sustains an E in unison with the violin's upper note throughout this excerpt, while the second violin and viola perform "combination tone" pitches based on the equations 2*a*-*b* and 2*b*-*a* (taking *a* as the partial number of the violins upper pitch and *b* as the partial number of the lower pitch).[67] For example, given the initial interval 15/11, the second violin plays 19 = 2(15)-11 and the viola plays 7 = 2(11)-15. The resultant intervals in the

[67] Note that this harmonic effect is closely related to frequency and ring modulation, both of which produce combination tones based on the interval between a carrier and modulator frequency. With his background in electronic music, Tenney would surely have been aware of such parallels. Both frequency and ring modulation have been the subject of harmonic experimentation by "spectral" composers and other European composers including Hans Zender, Peter Eötvös, and Clarence Barlow.

second violin and viola reflect the complexity of the generating interval in the first violin: thus, when the first violin reaches the perfect fourth B-E, the second violin and viola have the second and fifth partials of a consonant just E major triad. The resultant pitch sets of this algorithm, as Tenney points out, always can be understood as a subset of the partials of a single fundamental, which shifts as the first violin's pitches gradually change. The partial numbers for this excerpt (with associated cent values for each pitch class) are shown in Figure 1.24: the last row of the figure give the pitch class of the fundamental in cents and its registral position.

vn1 = *a*	15	*402*	27	*402*	4	*402*	33	*402*	17	*402*	13	*402*
vn1 = *b*	11	*1065*	20	*1082*	3	*1104*	25	*1121*	13	*1138*	10	*1147*
vn2 = 2*a*-*b*	19	*812*	34	*801*	5	*788*	41	*778*	21	*768*	16	*761*
vla = 2*b*-*a*	7	*283*	13	*337*	2	*402*	17	*454*	9	*501*	7	*530*
root		*514*		*696*		*402*		*349*		*297*		*761*
		F_1+14		**G_0-4**		**E_3+2**		**E-flat$_0$+49**		**E-flat$_1$-3**		**A-flat$_1$-39**

Figure 1.24: Partial numbers, cent values, and fundamentals of chords in Figure 1.23

Each of the four-note chords generated by the equations in this section of the piece is in the form x, x+y, x+2y, x+3y; that is, the difference between each adjacent chord member consists of the same number of partials. Tenney's choice of this spacing is likely to reflect an interest in the combination tone $2f_1$-f_2 (where f_1<f_2), identified by Guido Smoorenburg as the most audible of the combination tones.[68] Other composers, including Ezra Sims (see Chapter 3) have described harmonies arranged to fit collections of combination tones as particularly appealing to the ear.

The derivation of these complex harmonies from the two-pitch simultaneities of the solo violin *Koan* suggests how much is implied by a single interval. When we evaluate an

[68] Guido F. Smoorenburg, "Audibility Region of Combination Tones," *Journal of the Acoustical Society of America* 52/2 (1972): 603-614.

interval as a ratio, we can easily recognize its root implication (the "1" of the ratio) and from that root, predict an overtone series template that associates many additional pitches which can fit into the sonority without changing the root.

Premise 4: We recognize just intervals even when they are slightly mistuned.

Gérard Grisey: *Partiels* (1975)

The harmonic resources of the overtone series also fascinated composers of the French "spectral" movement, beginning in the early 1970s. This movement, led by composers Gérard Grisey and Tristan Murail, developed quite independently from the interest in the overtone series shown by American experimentalists like Tenney and Lucier—the spectralists' compositional influences were mainly European, including Giacinto Scelsi, Karlheinz Stockhausen (particularly his 1968 *Stimmung),* and Olivier Messiaen (who was a mentor to Grisey, Murail, and other members of the L'Itinéraire group).

At its inception, spectralism was seen as an antidote to the mathematical abstraction of Darmstadt serialism—though aspects of the serialist aesthetic have always occupied spectral composers, especially the division of sound into independent parameters and the use of schematic formal designs. This serialist legacy makes the aesthetic of the spectral composers quite different from that of Tenney and Lucier—perhaps the most obvious difference is the spectralists' preference for a much more active and dramatic sense of formal development.

Grisey's influential ensemble work *Partiels* begins with a grand gesture—the trombone's fortissimo blast on a low E is accompanied by vehement short downbows from the double bass. As the trombone fades away, the other instruments of the ensemble enter one by one with pitches corresponding to the partials of the trombone—the actual trombone sound is replaced by a carefully orchestrated replica, a stylized artificial "timbre" with complex instrumental sounds replacing each partial of the original (see Figure 1.25).

Figure 1.25: Instrumental synthesis of a trombone sound in Grisey's *Partiels* (1975)

Spectral composers called the scoring of the partials of a complex tone for instruments "instrumental synthesis" *(synthèse instrumentale).* The new availability of computer technology for sound analysis gave the spectralists a more complete understanding of how real-world sounds were built up of component partials—Figure 1.26 shows a spectral analysis of a trombone tone. This technology made it possible to calculate the relative amplitude of each of the partials, for example, and to examine the temporal evolution of a sound spectrum over time: in this trombone sound, the higher partials enter slightly later than the lower ones. Grisey uses this information in *Partiels* to assign dynamics to each of the instruments participating in the synthesis, and to shape the order of entries. In most brass sounds, the upper partials emerge slightly later than the lower ones, a phenomenon which Grisey imitates (on a much expanded time scale) with the staggered entries in his synthesized replica of the trombone.[69] This fascination with timbre is an essential part of the spectral aesthetic—spectral composers have been particularly drawn to the analogy between the partials which make up a complex sound and the pitches of an orchestrated harmony; instrumental synthesis makes this analogy audible, suggesting intriguing parallels between timbre and harmony.

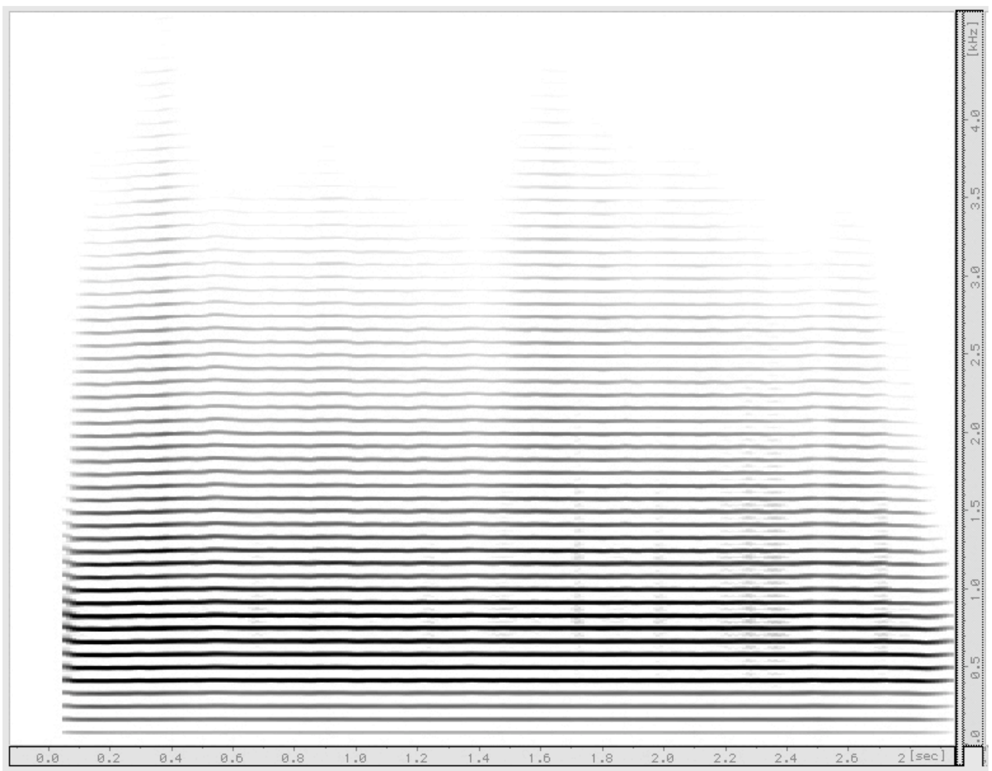

Figure 1.26: Spectral analysis of a trombone sound

It's instructive to compare the notation of Grisey's overtone chord to Tenney's more precise notation in *Koan for String Quartet.* By specifying both cents and ratios, Tenney provides an *exact* notation of the intervals of the overtone series, as well as a very "high resolution" approximation—the approximation to the nearest cent arguably provides more tuning detail than a human performer can hope to accurately realize. In

[69] It would be an oversimplification to associate the spectralists solely with the technique of instrumental synthesis, but particularly in the early days of the movement, this was an essential and frequently-used technique. Later developments in the spectralists' technique added a variety of effects and transformations, including many (including frequency modulation) based on tools of the electronic music studio.

contrast, Grisey retains standard equal-temperament tuning, supplementing the "normal" pitches with microtonal symbols indicating quartertones. In addition, he uses a down arrow for the seventh partial, which at 969 cents above the fundamental falls between the equal-temperament minor seventh of 1000 cents and the quartertone-flat minor seventh 950. With the exception of the seventh partial, all of the other pitches are rounded off to a quartertone grid. (The seventh partial falls between the pitches of the quartertone grid, and therefore requires a different notation if it is to be closely approximated.)

In contrast to Tenney's just intonation, then, Grisey's notation depicts an out-of-tune, approximated version of the overtone series. Our tolerance for mistuning, though, means that even a distorted and rounded-off version of the overtone series retains many of its essential harmonic qualities—we can still hear, for example, the "rootedness" of the sonority, or the relation of the partials to the fundamental, even if the partials are not exactly in tune.[70]

Grisey's quartertone equal temperament has precedents in the microtonal divisions of the octave pioneered by early twentieth-century composers like Hába and Wyschnegradsky. In an approach to microtonality very different from that of just intonation composers like Partch and Tenney, these composers took one acoustically tuned interval—the octave—and derived all of the other pitches of their gamut by the subdivision of the octave into equal parts, creating scales of quartertones, third tones, and sixth tones. This procedure imitated the division of the octave into twelve equal semitones which has become the standard keyboard temperament. While such equal

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[70] A psychoacoustic study by Brian Moore suggests that partials can be mistuned by as much as a quartertone without destroying the effect of "virtual pitch": Moore, Glasberg, and Peters, op. cit. While Tenney has argued for a degree of tolerance, his tolerance range is considerably smaller than that implied by Grisey's score—Tenney restricts mistuning to about 5 cents in either direction from the referential just interval.

divisions of the octave tend to favor a "distance" approach to interval, some microtonal composers exploring such extended equal temperaments were interested in the way that the new divisions could approximate just intervals—this led to some exotic divisions of the octave into 19 (Yasser) or 31 parts (Fokker). Grisey was also interested in replicating natural just intervals in the grid of quartertone (24-tone-per-octave) equal temperament—the fact that just intervals can be convincingly implied within a quartertone grid is a tribute to our ear's flexibility and tolerance for mistuning.

* * *

As discussed in Section 3, while Terhardt's algorithm for finding virtual pitch is based on precise subharmonics of each component pitch, a degree of flexibility or tolerance is accepted in the final determination of virtual pitch. Such flexibility is an absolute necessity in any theory that seeks to translate the abstractions and simple numerical relationships of just intonation into real-world sounds.

Some flexibility is required even in the simplest tonal progressions. Figure 1.27 repeats the just intonation *Tonnetz* of Figure 1.11. The C major scale indicated by bold note names allows the formation of five just triads: major triads on F, C, and G, and minor triads on A and E. The remaining diatonic triad of the scale, D minor, can only be approximated with this pitch set—its fifth, A, is calculated as the 5/4 third above F rather than as a perfect fifth from D, and is thus too low by 22 cents (a syntonic comma, or 81/80). If we raise the A to a perfect fifth above D, then the F and A triads are out of tune by the same amount.

F#	C#	G#	D#	A#	E#	B#	Fx
568	*70*	*772*	*274*	*976*	*478*	*1180*	*682*
D	**A**	**E**	**B**	F#	C#	G#	D#
182	*884*	*386*	*1088*	*590*	*92*	*794*	*296*
B♭	**F**	**C**	**G**	**D**	A	E	B
996	*498*	*0*	*702*	*204*	*906*	*408*	*1110*
G♭	D♭	A♭	E♭	B♭	F	C	G
610	*112*	*814*	*316*	*1018*	*520*	*22*	*724*

Figure 1.27: Just intonation *Tonnetz*

Many solutions have been proposed for this problem, including keyboards with extra keys to accommodate different versions of certain notes; theorists Moritz Hauptmann (1792-1868) and Simon Sechter (1788-1867) dispensed with the problem by denying that the 2-6 fifth is perfect, describing it instead as diminished. The most enduring solution, though, have been various kinds of *temperaments*—each interval is made slightly out of tune with regard to the ideal just interval, but no (or few) intervals are so out of tune that they are unusable. In current practice, the standard temperament is equal temperament—every step on the keyboard is made the same size, exactly one-twelfth of an octave. This temperament makes fifths only 2 cents (hundredths of an equally-tempered semitone) out of tune, but unfortunately makes each major third 14 cents sharp. As a result, the equally-tempered third is harsher and more tense than the smooth just third. Historically, from the eighteenth to the twentieth century, the flexibility of equal temperament outweighed the purity of sound of just intonation.

Tempered pitch systems use a limited number of pitch classes, in contrast to the potentially infinite number of pitch-classes in just intonation. To bring the infinitude of just intervals into a finite set, the just intervals must be slightly mistuned. An example in twelve-tone equal temperament is the "circle of fifths"—the idea that a series of rising fifths will eventually "loop around," returning to the starting point in pitch-class space. If the fifths are just, though, the series of fifths never closes—we arrive at something more like a spiral than a circle. At the point where the equal temperament circle of fifths would close in on its origin, the just intonation spiral of fifths is too sharp by a Pythagorean comma of about 24 cents.

Equal temperament shrinks the fifth slightly, from about 702 cents to exactly 700 cents, which allows the circle to close. (While this might seem like hair-splitting, the difference between tuning pure fifths and tempered fifths is a practical matter for string players who tune in fifths, especially when they need to match the fixed equal-temperament tuning of the piano.) The centrality of the triad to Western harmony means that a system of temperament must consider thirds and sixths as well as fifths and fourths. Here, equal temperament leads to much greater mistunings of the just intervals—the 5/4 major third, for example, is approximately 386 cents in just intonation, but 400 cents in equal temperament. The difference of 14 cents is clearly audible—performers will often adjust thirds downward, closer to the just interval. In practice, then, ensemble tuning is a constant compromise, between the desire to play in just intonation and the flexibility of harmonic motion offered by equal temperament.

Temperament might have originated as a practical solution to everyday problems of tuning, but it exposed some serious contradictions in the foundations of music theory. If out-of-tune versions of the ideal just intervals are musically acceptable, why bother with the ratios at all? And if we've been using ratios to define consonance, what does it mean that the 5/4 ratio of a major third can be replaced in equal temperament by the *irrational* number $\sqrt[3]{2}/1$?

To explain these problems, we must recognize that the properties of just intervals discussed so far: their harmonic referentiality, rootedness, and harmonic implication, persist even when the just interval is slightly mistuned. The practice of temperament indicates that this has long been accepted as a tacit theoretical principle, but recent scholarship has expanded on its psychological and musical implications.

A psychological concept with clear application here is the idea of categorical perception—in certain sensory domains, we do not experience a smooth continuum of different values, but rather find that phenomena are grouped into discrete, qualitatively different categories. We noticed this phenomenon in Tenney's *Koan* for solo violin—despite the even expansions and contractions of the violin's interval, certain intervals could be identified as substantially different in quality from their neighbors. In linguistic research, where the theory first emerged, categorical perception describes an unusual experimental effect: in a series of vocal sounds carefully graded from "*b*" to "*p*," subjects do not recognize a continuum, but rather only "*b*s" or "*p*s"—nothing in between.[71] The musical applications are obvious, and as Edward Burns points out, categorical perception simply offers an empirical verification of what has been recognized for centuries: that "there is considerable latitude allowed in the tuning of intervals that will still be considered acceptable and will still carry the appropriate melodic information."[72]

While it is likely that the specific categories of our pitch perception are at least in part culturally determined, I have argued that just intervals represent important referential points for harmonic perception (see Premise 1 above). One of the most articulate

[71] See Burns and Ward, op. cit., 226. The first research in categorical perception was published by A. M. Liberman, K. S. Harris, H. S. Hoffman, and B. C. Griffith, "The discrimination of speech sounds within and across phoneme boundaries." *Journal of Experimental Psychology* 54 (1957): 358-368.

[72] Burns and Ward, op. cit., 231.

advocates of this viewpoint in recent years has been James Tenney: unlike the “purist” just intonation composers Harry Partch and Ben Johnston, Tenney believes that we understand the harmonic identity of a just interval even when that interval is not perfectly in tune.

> Now, I propose as a general hypothesis in this regard that the auditory system would tend to interpret any given interval as thus ‘representing’—or being a variant of—*the simplest interval within the tolerance range* around the interval actually heard (where “simplest interval” means the interval defined by a frequency ratio requiring the smallest integers). The simpler just ratios thus become “referential” for the auditory system . . . Another hypothesis might be added here, which seems to follow from the first one, and may help to clarify it; within the tolerance range, a mistuned interval will still carry *the same harmonic sense* as the accurately tuned interval does, although its timbral quality will be different—less “clear,” or “transparent,” for example, or more “harsh,” “tense,” or “unstable,” etc.[73]

The range of tolerance, Tenney suggests, would be inversely related to the complexity of a ratio—we can accept an octave despite considerable retuning, but a complex ratio like 19/16 needs precise tuning if it is to be recognized. Among the many possible just ratios approximating any given pitch interval, we tend to choose the simplest one: this indicates a perceptual bias toward simple intervals, since very complex ratios are likely to be heard as mistuned versions of simpler ratios.[74] Tolerance makes it possible for the quartertone approximations at the beginning of Grisey’s *Partiels* to imply in-tune just intervals. If we accept still greater degrees of tolerance—approximations to a coarser “grid”—we will

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[73] Tenney, “The Several Dimensions of Pitch,” 110 (emphasis in original).

[74] This idea closely parallels Hugo Riemann’s “principle of the greatest possible economy for the musical imagination,” discussed in detail in Chapter 2.

find that complex extended just intonation chords can be implied even in the twelve-tone equal temperament of a piano.[75]

Alexander Scriabin: *Poème,* op. 69/1 (1912-13)

A strength of a pragmatic theory of harmonic perception is its applicability to music of very different styles and periods. We can apply the same harmonic principles demonstrated above in Tenney's *Koan for String Quartet* to passages in Scriabin's enigmatic late works. Figure 1.28 shows the first eight measures of Scriabin's *Poème,* op. 69/1. Throughout this passage, the harmony is based largely on a single whole-tone scale, which gives rise to four distinct harmonies, one for each two-bar unit. The first and third of these are transpositions of Scriabin's so-called "mystic chord," while the second and fourth are closely-related sets. Though these are not traditional tonal sonorities, to my ear each one still has a distinct sense of root—on C, A-flat, E, then A-sharp—despite the largely overlapping pitch-class content. How do these complex harmonies relate to the heard roots?

[75] Célestin Deliège has also explored extended just intonation readings of music in twelve-tone equal temperament: see "L'harmonie atonale: de l'ensemble à l'échelle" in *Sources et ressources d'analyses musicales: journal d'une démarche* (Sprimont: Mardaga, 2005).

Figure 1.28: Alexander Scriabin, Poème, op. 69/1 (1912-13): mm. 1-8

To answer this question, we can return to Leonid Sabaneyeff's 1912 description of Scriabin's mystic chord as derived from the overtones of its bass note—as "the seventh through the thirteenth partials omitting the twelfth."[76] This correspondence is illustrated in Figure 1.29.

"Mystic chord"	*ET cents*	partial number	*JI cents*	difference between JI and ET
A	*900*	13th partial	*841*	-61
F-sharp	*600*	11th partial	*551*	-49
E	*400*	10th partial	*386*	-14
D	*200*	9th partial	*204*	+4
C	*0*	8th partial	*0*	0
B-flat	*1000*	7th partial	*969*	-31

Figure 1.29: Interpretation of Scriabin's "mystic chord" as a collection of partials; comparison of equal tempered (ET) and just intonation (JI) versions

Certain overtones are difficult to realize on the piano, where their nearest approximation is necessarily out-of-tune. This is particularly true for the 11th and 13th partials, which at 551 and 841 cents above the fundamental fall almost exactly between two equal temperament intervals. The 11th partial needs to be rounded upward to a tritone

[76] Clifton Callender, "Voice Leading Parsimony in the Music of Alexander Scriabin," *Journal of Music Theory* 42/2 (1998): 219-233. See also Richard Taruskin, *Defining Russia Musically: Historical and Hermeneutical Essays* (Princeton, New Jersey: Princeton University Press, 1997): 342.

above the root—if it is rounded downward, the resultant interval is immediately recognized as a perfect fourth instead of an eleventh partial. Though the thirteenth overtone is slightly closer to the minor than the major sixth, either equal-temperament interval can make a convincing approximation of this overtone, depending on the context. When the 11^{th} overtone is rounded from 551 to 600, the interval 13/11 is best preserved by rounding the 13^{th} overtone up as well, from 841 to 900. When the 13^{th} overtone is rounded up to 900 cents, it can also be interpreted as a version of the 27^{th} overtone (906 cents). (Though 27 might seem like a much more complex representation, its reducibility to 3×3×3 makes it simpler, in one sense, than the irreducible prime number 13.) Other interpretations of the interval are as a just major (5/3) or minor (8/5) sixth, though these would each imply a different root than the one suggested here. Sabaneyeff's description has been much disparaged by recent scholars as unacceptably vague and neglectful of the chord's tonal implications. In this context though, tonal implications are constantly defeated as the harmony drifts from one chord to the next—Sabaneyeff's description, expanded and clarified by Tenney's principle that we choose the simplest just interval near each heard interval, can help to interpret this harmonic progression. My analysis of the passage is illustrated in Figure 1.30.

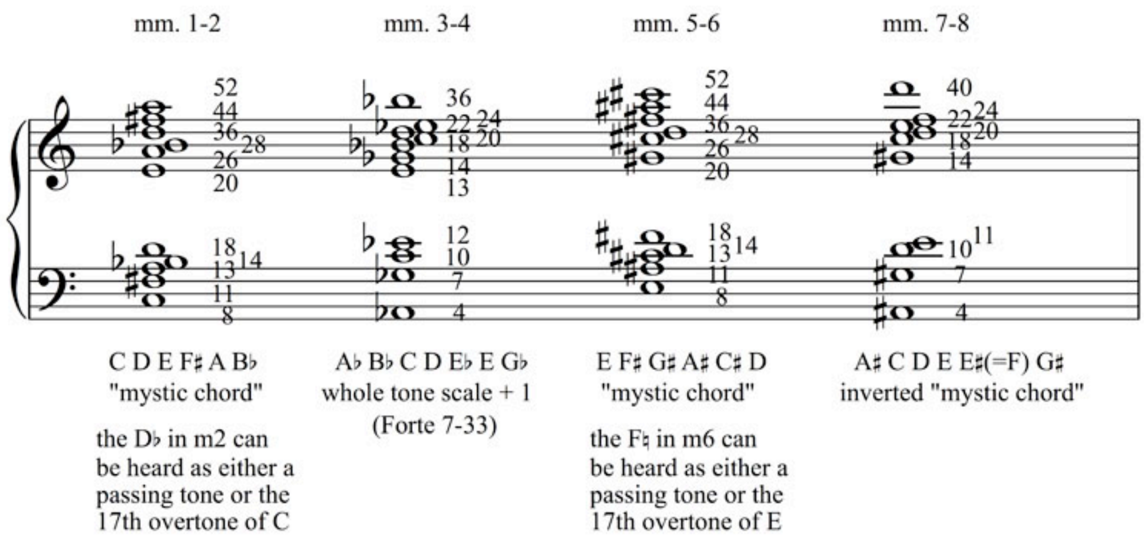

Figure 1.30: Analysis of mm. 1-8

As the passage progresses, pitches of this opening chord are reinterpreted—the pitch D, for example, is heard first as the 9th overtone of C, then the 11th of A-flat, 7th of E, and 5th of A#. The shifting harmonic meanings caused by the changing fundamentals reflect the mutable, floating sense of tonality in this passage. We can see the importance of register in determining harmonic meaning—in many cases, nearly identical whole-tone material is given a different harmonic sense by a change in the bass note. This is due not only to the symmetry of the whole-tone scale, but also to the near correspondence of its pitches to overtones 7 to 13 of several different fundamentals. Considering these pitches as overtones allows us to talk specifically about the importance of register in determining roots, and also to describe the internal relationships between chord tones, including their relative harmonic distances and degree of relatedness to the fundamental and one another. Aspects of ratio theory might also be used in combination with more traditional modes of analysis—see for example Olli Väisälä's 2002 article which applies Schenkerian prolongational techniques to overtone-based chords in music by Scriabin and other early

20th-century composers including Berg and Debussy.[77] (This echoes Ben Johnston's contention that Debussy's harmonic language "approximates as well as can be in equal temperament a movement from overtone series to overtone series, with an emphasis on higher partials."[78])

* * *

My analysis of Scriabin's *Poème* is indebted to Väisälä's important research, which deserves further consideration here. He notes that while theories based on the overtone series were important to composers like Schoenberg and Hindemith, such theories seem to have fallen out of favor in recent years—his paper proposes a "qualified rehabilitation." Väisälä's analyses examine the way that a chord based on the overtone series—approximated on the equal-temperament keyboard—can be the underlying harmony for a Schenker-style prolongation. This hybrid approach is a fascinating departure from the pitch-class set approach that has dominated work on the music of these composers. His identification of overtone-based chords in Scriabin's late piano piece *Vers la Flamme,* Op 72 (" toward the flame") demonstrates the possibility that the ratio-model of pitch might be used to describe pitch structure in "atonal," tempered music.

Väisälä is not the first theorist to discuss overtone collections in Scriabin's music, as we already saw in Sabaneyeff's analysis of the mystic chord. Väisälä is cautious to avoid basing his theory on the idea that the overtone harmonies he uncovers were the composer's explicit intention—his theory works whether or not the harmonies were

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[77] Olli Väisäla, "Prolongation of Harmonies Related to the Overtone Series in Early-Post-Tonal Music," *Journal of Music Theory* 46/1-2 (2002): 207-283.

[78] Ben Johnston, "A.S.U.C. Keynote Address," *Perspectives of New Music* 26/1 (Winter, 1988): 236.

intentionally constructed to match the overtone series—but it seems quite likely that Scriabin was aware of and interested in overtones. The phenomenon of the overtone series has often attracted composers of a mystical sensibility—their combination of otherworldliness and naturalness meshes well with a mystical worldview, as does the almost meditative experience of immersion in the overtones of a sound (see for example the music of Giacinto Scelsi or Dane Rudhyar).

Väisälä's article can be read as a critique of set theory: in his conclusion, Väisälä identifies "octave equivalence and the treatment of other intervals in terms of equal divisions of the octave" as the two premises underlying set theory. Having argued elsewhere for an expanded consideration of register and the rejection octave equivalence in the context of certain atonal works,[79] Väisälä questions here the second premise, the idea that other intervals should be primarily understood as divisions of the octave (i.e., distances measured in equal-tempered semitones):

> While this premise is adequate for generating the equal-tempered pitch space, it is less adequate for the description of the perceptually and structurally significant aspects that emerge in music employing that collection. Discussing some of the most significant music from the advent of post-tonal music from the advent of post-tonal harmony, I have attempted to show that its organization is illuminated by allowing for the root-supporting status of intervals, a property that does not stem from their width in fractions of the octave but from their correspondence with the intervals in the harmonic series.[80]

By measuring all intervals as "fractions of the octave," set theory assumes an essentially "flat" tonal space, where except for the octave, intervals differ in size but not in quality. To account for the "root-supporting status of intervals," though, Väisälä needs

[79] Olli Väisäla, "Concepts of Harmony and Prolongation in Schoenberg's Op. 19/2," *Music Theory Spectrum* 21/2 (1999): 230-259.

[80] Olli Väisäla, "Prolongation of Harmonies Related to the Overtone Series in Early-Post-Tonal Music," 271.

to turn to the harmonic series, and the way that acoustic and psychoacoustic considerations create privileged intervals in the pitch continuum. The result is a space that is no longer flat, but filled with landmarks: the just intervals stemming from the overtone series.

On the basis of the psychoacoustical theories of Ernst Terhardt, Väisälä adopts a theory of rootedness: "If a bass tone forms similar intervals with the upper tones as those between a fundamental and its harmonics, it will have a tendency to be perceived as the root of the harmony—that is, governing the overall pitch pattern in the manner of a virtual pitch." After Parncutt's 1988 extensions to Terhardt's theories, Väisälä proposes a list of "root supports" based on the partials 1 to 10: "from the strongest to the weakest: octave/unison, P5, M3, m7, M2."[81] Root supports are calculated like figured-bass intervals, from a bass note. Väisälä adds the tritone (approximating the eleventh partial) as an additional, if weaker root support. Väisälä's analyses combine this model of additional root supports with Schenkerian prolongational procedures, prolonging overtone series harmonies including any or all of the root supports instead of the standard major/minor triad.

Väisälä's theory is based on an assumption that approximations of just intervals offer the same kind of root support as their purely-tuned neighbors—this assumption allows the interaction of just intonation ideas with the vast body of music conceived for the twelve note per octave scale of the piano. Such an interaction allows analytical associations impossible in canonical pitch-class set theory—one of the most useful tools is the ability to associate sets based on their similar qualities of rootedness rather than on shared membership in an abstract set class. By focusing our attention on a different kind

[81] Ibid, 209-210.

of pitch relation, this approach may illuminate aspects of musical experience that would otherwise escape notice. One promising feature of an approach which recognizes approximate just intervals in tempered settings is a renewed attention to register—the spacing of a chord can have a major effect on how its rootedness (or lack thereof) is perceived. We might also consider how temperament creates ambiguity between different just interpretations—the possibility for multivalence or *Mehrdeutigkeit* has been one of the historical advantages of tempered tunings. Of course, not all music written in equal temperament will benefit from a consideration of approximated just intervals or overtone series (a Babbitt piano piece, for example, seems less likely than a Debussy prelude to have useful root associations)—but, as I set out to show here, the inclusion of such a theory in our analytical toolkit may still provide useful insights in some unexpected contexts.[82]

In the music we've explored so far, we've been able to experience harmonic rootedness in fairly unambiguous contexts: pitches have been reducible to members of a single root at a time. But music—especially the complex repertoire of the twentieth-century—rarely presents us so clearly with chords that are comprehensible as selections from a single overtone series. Can the theoretical premises advanced so far offer ways to decipher more ambiguous harmonies?

[82] The association of tempered intervals with nearby just intervals does not rule out the possibility that many sonorities in tempered music are chosen precisely for their sound when played *as tempered intervals;* recognizing the harmonic interpretation offered by just intonation does not require ignoring the actual tempered sonority. Rather, both aspects of our experience of the sonority can successfully coexist.

Premise 5: Faced with large and complex harmonies, we resolve them into combinations of multiple simpler harmonies if possible.

The fifth premise of my argument addresses the *segmentation* of aural data into separate entities, either single or compound in nature. Here we shall see the influence of recent studies in psychoacoustics, especially Albert Bregman's *Auditory Stream Segregation.* Perception, in Bregman's definition, is "the process of using the information provided by our senses to form mental *representations* of the world around us." His central theme, "auditory scene analysis" is a phrase coined by analogy to visual scene analysis. Bregman's example of a scene analysis problem in vision is our parsing of a line drawing of overlapping blocks. To understand the drawing, we have to determine which areas of the drawing belong together—if we perform this incorrectly, the image will be "chimerical," as we mistakenly combine parts of different object which do not actually belong together.

A similar problem arises when we hear a sound in a noisy room. As Bregman writes, "A friend's voice has the same perceived timbre in a quiet room as at a cocktail party. Yet at the party, the set of frequency components arising from that voice is mixed at the listener's ear with frequency components from other sources." If we mixed all of these sources together, as if they came from a single source, the timbre of our friend's voice would be distorted by the foreign frequencies; but instead, we are able to separate our friend's voice from its surroundings. This is an instance of auditory scene analysis with obvious parallels to visual scene analysis.[83]

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[83] In two articles, Alfred Cramer has adapted aspects of Bregman's auditory scene analysis to atonal music by Schoenberg and Webern. In his 2002 *Music Theory Spectrum* article, Cramer sketches a system of graphic notation which diagrams different kinds of fusion, sequential linkage, and hierarchical relationships between tones. One of his most interesting claims is that the altered octaves (octaves plus or minus a semitone) common in this music create an effect of "unpitched fusion," in which the emergent tone color of

Our ears are never presented with completely isolated sonic objects—rather, we receive a complex signal of mixed sounds. John Cage has noted that even in the complete sonic isolation of an anechoic chamber, true silence is not heard—the listener's own physical processes produce an unavoidable background noise.[84] Our ability to separate this jumbled input into separate sources is an astonishing feat of cognitive processing. Many of our stream-segregation procedures are reminiscent of the grouping rules of *Gestalt* theory, which finds a music-theoretical expression in James Tenney's *Meta-Hodos.* For example, Bregman suggests that we apply an "old-plus-new heuristic": "If you can plausibly interpret any part of a current group of acoustic components as a continuation of a sound that just occurred, do so and remove it from the mixture. Then take the difference between the current sound and the previous sound as the new group to be analyzed."[85]

Particularly remarkable is our ability to separate a texture of multiple complex pitches into the sounds of individual instruments, keeping their distinctive timbres separate. As a basic spectrogram would show, the sound of many instruments playing at once gives us a number of separate partials—the trick is to group together the partials which emanate from the same source so that we can separate the sounds of individual instruments. One of the most important strategies is to group together all of the partials

the major seventh or minor ninth is more important than the actual pitch of either component of the interval. The fusion, Cramer argues, comes not from the consonance of these intervals (which is relatively low), but rather from the high degree of masking between their partials, which increases the difficulty of resolving clear individual pitches. See "Schoenberg's 'Klangfarbenmelodie': A Principle of Early Atonal Harmony," *Music Theory Spectrum* 24/1 (2002): 1-34 and "The Harmonic Function of the Altered Octave in Early Atonal Music of Schoenberg and Webern: Demonstrations Using Auditory Streaming," *Music Theory Online* 9/2 (2003), http://mto.societymusictheory.org/issues/mto.03.9.2/mto.03.9.2.cramer_frames.html (accessed April 15, 2008).

[84] John Cage, *Silence* (Middletown, Conn.: Wesleyan University Press, 1961): 8.

[85] Bregman, op. cit., 222. Note that this kind of loose guideline resembles my "preference rules," as introduced in Chapter 2.

which belong to the same overtone series, and are thus likely to come from the same source. (This reflects our experience that many real-world sounds have a harmonic partial structure.) As Bregman notes, we seem to apply "a scene-analysis mechanism that is trying to group the partials into families of harmonics that are each based on a common fundamental. If the right relations hold between an ensemble of partials, they will be grouped into a single higher-order organization."[86]

Bregman goes on to describe the cognitive processing of multiple complex tones heard simultaneously:

> If the two fundamental frequencies are unrelated, an analysis that tries to find a single fundamental for all the partials that are present will fail. Yet we know through listening to music that we can hear two or more pitches at the same time. There has to be a way to base the computation of each component pitch on only a subset of the partials. Assuming that this is somehow done, what we hear in the presence of a mixture of two harmonic series (as when two musical tones are played at the same time) is not a large set of partials but two unitary sounds with distinct pitches.[87]

Older theories of pitch perception assumed that we "process" only one complex tone at a time: all the signals from the ear were thought to go somewhere in the brain where the best-fitting fundamental would be adduced. Bregman's theory explains how we can separate out the sensory input into individual complex tones through a variety of segregation mechanisms. When we apply these principles to musical analysis, we will typically be interested in phenomena at the higher level of complex tones (musical notes) rather than the lower level of individual partials. Many of the musical experiments of the spectral school suggest that such a transfer of principles from one level to another is plausible and aurally convincing. Moreover, we can use these principles to explain many

[86] Bregman, op. cit., 507.

[87] Bregman, op. cit., 233. See also Hartmann, op. cit., Chapter 6.

aspects of common practice tonality and voice leading.[88] One could even loosely describe consonance and dissonance in tonal music as representing a contrast between belonging and not belonging in a given harmonic grouping. As we shall see, these principles can also shed light on "atonal" works, where the complexity of harmonic relationships often creates intriguing ambiguities of harmonic grouping and rootedness.

Arnold Schoenberg: *Erwartung,* op. 17 (1909)

The application of some of these concepts of "auditory stream segregation" can be demonstrated in an excerpt from Schoenberg's monodrama *Erwartung.* In his theoretical writing, Schoenberg often invoked the idea that complex and dissonant harmonies might be understood as upper overtones. While in many cases this idea seems to be only a metaphor for the increased dissonance in his atonal and serial music, we can also take it literally, seeking ways that overtone formations might be "segregated" into separate harmonic entities in Schoenberg's harmonies.

In the chapter titled "Aesthetic Evaluation of Chords with Six or More Tones" in his *Theory of Harmony,* Schoenberg cites a chord of thirteen notes—and eleven different pitch classes—from the fourth scene of *Erwartung,* just before the break of dawn (measure 382). See the reduction of the chord to two staves in Figure 1.31, top system.[89] Schoenberg notes that in chords like this one, the dissonances can often be softened by

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[88] See David Huron, "Tone and Voice: A Derivation of the Rules of Voice-leading from Perceptual Principles," *Music Perception* 19/1 (2001): 1-64. In particular, note Huron's "Tonal Fusion Principle": "The perceptual independence of concurrent tones is weakened when their pitch relations promote tonal fusion. Intervals that promote tonal fusion include (in decreasing order): unisons, octaves, perfect fifths, ... Where the goal is the perceptual independence of concurrent sounds, intervals ought to be shunned in direct proportion to the degree to which they promote tonal fusion" (18-19).

[89] Some published versions of *Erwartung* include a slightly different version of this chord—the version in the complete works matches the example in the *Theory of Harmony* discussed here.

"wide spacing of the individual chord tones." This makes them more readily comprehensible, he suggests, by clarifying their origin as "remote overtones."

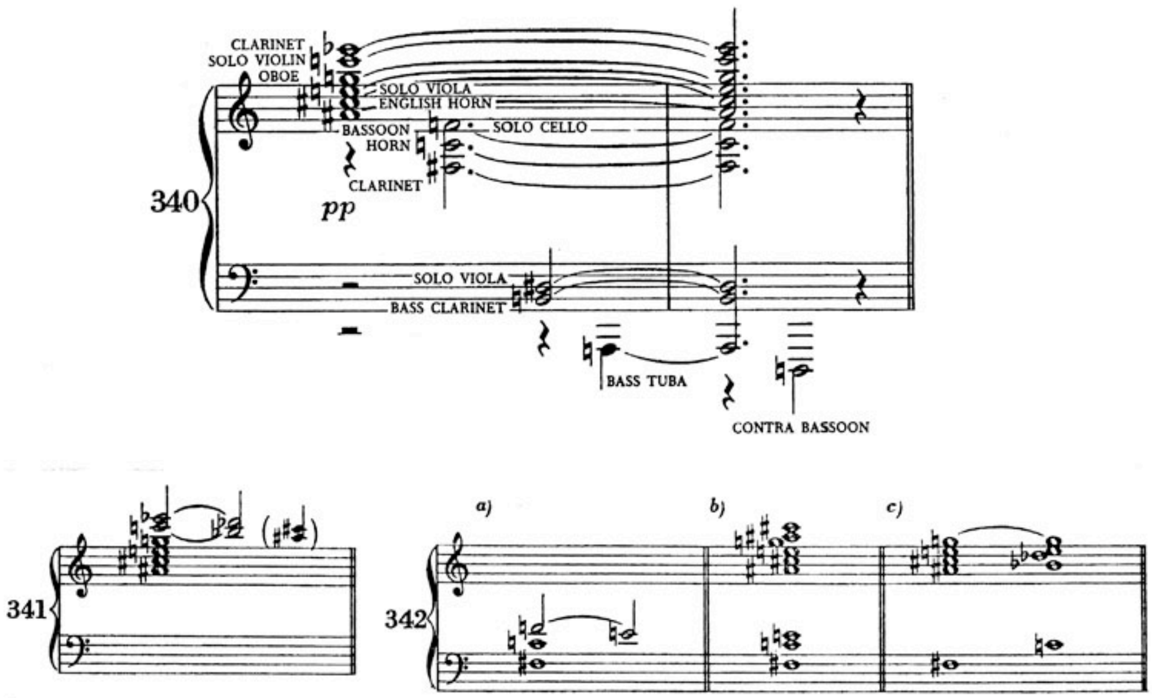

Figure 1.31: Three figures from Schoenberg's Theory of Harmony (1911, rev. 1922)

We can recognize the difficulty of an analysis of this chord through conventional techniques—pitch class set theory, for example, can say little about a sonority with eleven different pitch classes, and also ignores register, which seems to be of prime importance here. Schoenberg's approach, illustrated in Figure 1.31, seeks first to reduce the number of pitches by explaining away certain tones. In the figure labeled 341, Schoenberg shows that he hears the C and E-flat as dissonances with expected resolutions to B-flat and D-flat, pitches already present in the chord. 342a indicates a similar resolution from F to E. Though the pitches never reach their expected destination in the excerpt, Schoenberg argues that their harmonic meaning as unresolved upper neighbors is nonetheless clear. By replacing these tones with their resolutions, Schoenberg arrives at the simpler chord in 342b, which he believes can be resolved into two superimposed minor ninth chords sharing a diminished seventh chord—see 342c. Schoenberg's conclusion is that the chord is comprehensible because it can be referred to more traditional and familiar pitch combinations.

Schoenberg's analysis trails off here, leaving the cello's F and the four final pitches in the bass clef unexplained. In my Figure 1.32, I offer an alternative analysis based on similar principles, but hewing more closely to the implications of psychoacoustic research.

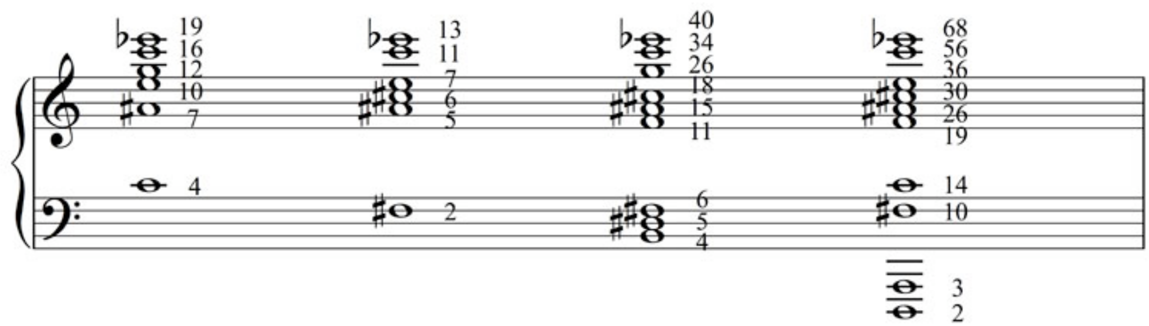

Figure 1.32: An alternative analysis

Here, I've reinstated the notes that Schoenberg explains away as unresolved neighbors. My analysis follows Schoenberg's in explaining the chord on the second beat as a combination of sonorities on C and F#—though I've identified overtone structures over each root in place of Schoenberg's minor ninth chords. This follows from the stream segregation principle which states that one groups together pitches that can be mapped to a single fundamental. Several pitches can be heard as belonging to either root. In my reading, the F natural lacks a clear harmonic meaning until the arrival of the B and D# on beat 3—these imply the root B, with the F as an 11th partial. Alternate harmonic meanings for the F#, C, and F natural arise with the bass fifth D-A—they can now be heard as representing partials 10, 14, and 19 of D.

The chord shows the pull of several different potential roots, which allow the notes in the upper register to be interpreted in several different ways. This ambiguity reflects the sonic complexity of the musical surface.[90] Note that this analysis is not atonal, but rather tonal, in an extended sense of the word—every pitch can be interpreted as harmonically related to one or more roots. This approach reflects Schoenberg's own preference for the term "pantonal" rather "atonal" as a description of his music.[91]

Segmentation has always been a central topic in the analytic literature surrounding atonal music: deciding which sonic events belong together is the first step in making a pitch-class-set analysis. In most pitch-class-set analyses, segmentation is closely linked to an organicist aesthetic, which seeks to illuminate the music by showing motivic similarities. As a result, the segmentation is frequently made to reflect possible motivic relationships, with psychological principles (like Tenney's Gestalt rules of proximity, etc.) playing only a supporting role. If, as Bregman's research suggests, there is also a tendency to group pitches according to overtone-series templates, we can add a new principle of segmentation, based on the nature of our perception rather than a particular aesthetic bias toward motivic development. Segmentation by grouping pitches belonging

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[90] Bregman proposes a "Principle of Exclusive Allocation"—that ideally, perception tends to attribute any feature to only one object, not more than one. "A sensory element should not be used in more than one description at a time" (Bregman, op. cit., 12). In my discussion of multiple fundamentals in this excerpt, I've frequently chosen to flaunt this principle and allow single pitches to simultaneously relate to multiple roots. While such an allocation might be unusual in natural sonic environments, music frequently creates "chimeric" perceptions, fusing sounds from different sources into a single entity. As Bregman notes, "If a group of components have arisen from the same physical event, they will have relationships between them that are unlikely to have occurred by chance" (ibid., 221). Music frequently sets up these unlikely relationships between different sources—for example, two instruments playing in unison (or at the octave, fifth, etc.) share common frequencies and contours, which generally would imply a single source. This allows our perceptual mechanisms to hear a fused sound in certain circumstances. Especially in tonal harmonic music, such fusion between disparate sources is common as an organizational device.

[91] Schoenberg, op. cit., 432n.

to the same overtone series can be useful in the analysis of music in a variety of styles, including tonal, microtonal, and atonal works.[92]

Summary

The five premises introduced in this chapter are the basis for the theory of harmonic perception developed in Chapter 2. To briefly review:

1. Just intervals are the referential intervals for harmonic perception.

Within the continuum of pitch intervals, we find certain privileged intervals—these are clearly audible in the smooth traversal of interval sizes in Tenney's *Koan.* These privileged intervals are closely linked to the structure of the *partials* of a complex tone—the relationship between partials and the just intervals becomes evident in the *xöömei* tradition, represented here by Fedor Tau's recording *Steppe Kargiraa.*

2. The principles of standard just intonation can be extended to include higher prime numbers.

One of the most significant moments in the history of ratio-based interval theory was the expansion from Pythagorean (3-limit) to just (5-limit) tunings in the fifteenth century. The extension of harmonic resources to include higher prime numbers is in theory infinite—composers like Harry Partch and Ben Johnston have made significant contributions to extended just intonation, up to and beyond the 11-limit. The pitch

[92] Segmentation of a vertical sonority into separate parts is often cited as an arbitrary feature of pitch-class set analyses—why, critics ask, should a sonority attacked at the same time be split into more than one entity? Harmonic templates based on the overtone series offer a reasonable way of making vertical segmentations with the support of a convincing theoretical model.

language of extended just intonation is illustrated in a simple, monophonic line in Ligeti's Viola Sonata. La Monte Young's *The Well-Tuned Piano* shows how an extension to the 7-limit affects a familiar theoretical construct, the *Tonnetz.*

3. Each just interval implies a fundamental or root, and a specific closely-related harmonic domain based on the overtones of that fundamental.

Just intervals are more than simply consonances—they strongly imply a well-defined harmonic context based on the overtone series in which they can be found. We make judgments of potential harmonic roots based on interval size. What's more, each interval affects how we understand other points in the pitch continuum, depending on how they fit into the overtone series implied by that interval. This place-finding quality of intervals suggests that musical listening involves the constant projection of harmonic templates, which are confirmed or denied as the music unfolds, with dramatic effects on our interpretation and listening experience. We can hear a shift of harmonic template in an excerpt from Alvin Lucier's *Music on a Long Thin Wire.* In the string quartet version of James Tenney's *Koan,* the second violin, viola, and cello add pitches to the solo violin work that reinforce the harmonic implications of each interval.

4. We recognize just intervals even when they are slightly mistuned.

Following the precepts of just intonation to the letter can lead to an overwhelming number of pitches, and also to intonational difficulties in tonal music. The historical solution has been *temperament,* the slight mistuning of the just intervals to allow single pitches to fulfill multiple harmonic roles. Demonstrations of quartertone temperament

(Grisey's *Partiels)* and twelve-tone equal temperament (Scriabin's *Poème* Op. 69, 1) show how implications of extended just intonation can be conveyed even when played in tempered tunings.

5. Faced with large and complex harmonies, we tend to resolve them into combinations of multiple simpler harmonies if possible.

Just as we can sort aural signals into individual complex tones and timbres, by grouping together harmonically related partials, we tend to group together pitches in complex textures that have harmonic relationships with one another. We are capable of simultaneously maintaining multiple harmonic templates, which are often invoked by complex pitch structures, as demonstrated in an excerpt from Schoenberg's *Erwartung.*

The premises introduced in this chapter are the underpinning for the theory of tone representation developed in Chapter 2. The historical roots of extended just intonation are examined in Chapter 3, along with compositional, analytical, and speculative applications of just intonation theory. Chapter 4 is an analytical application of my theory of tone representation to Gérard Grisey's 1996 chamber work *Vortex Temporum*.

CHAPTER 2: A Theory of Tone Representation

Introduction: Toward a theory of tone representation

As established in Chapter 1, our auditory perception is naturally biased toward just intervals, the intervals found between the partials of the overtone series. The composer and theorist James Tenney has called the overtone series and the just intervals it contains the only perceptual "givens" in our understanding of pitch.[1] Just intervals are the historical basis of Western music theory: octaves, fifths and fourths, thirds and sixths are all based on simple just intervals whose frequency ratios can be expressed with the prime factors 2, 3, and 5. The just intervals are referential sonorities, in the sense that we understand them as the ideal versions of intervals, even when the intervals we actually hear are out-of-tune. Our tolerance for mistuned just intervals is evident in the historical development of temperaments: the essential harmonic meaning of the just interval remains, even when it is heard only in an approximate, tempered version.

When we match a heard interval to a referential just interval, we produce two essential pieces of data: the ratio relating the two pitches, and an implied root or fundamental: the "1" of the interval ratio. Given the pitches E_4 and G_4, for example, we identify both a just interval between the two (6/5), and the implied fundamental, C_2. The number assigned to a pitch imparts a harmonic meaning—in this example, the "5" means that we hear the E as the fifth partial of C, not (for example) as an independent root. The process of matching a given collection of pitches to a just-intonation interpretation is very similar to Hugo Riemann's concept of "Tonvorstellung," or "tone representation."

[1] James Tenney and Donnacha Dennehy, "Interview with James Tenney," *Contemporary Music Review* 27/1 (2008): 87.

Riemann proposes that the harmonic meaning of a tone is determined by how we "imagine" it as one of the components of a major or minor triad.

In Riemann's triadic model of tone representation, we understand all intervals within the framework of Renaissance just intonation, based solely on multiples of 2, 3, and 5—but we can expand the theory to allow more complex interval ratios with higher prime factors. This brings us into the harmonic world of "extended just intonation," developed by American experimental composers Harry Partch, Lou Harrison, and Ben Johnston. Extended just intonation includes many microtonal intervals which fall "between the keys" of twelve-tone equal temperament, such as the flat minor seventh (7/4, or 969 cents) or the undecimal tritone (11/8, or 551 cents). If we accept that approximations of these extended just intervals still convey the same harmonic meaning, many "atonal" sonorities of music of the twentieth century can be understood as equal-temperament approximations of pitch collections in extended just intonation. Schoenberg himself suggested that the future of musical "evolution" would rest on "the growing ability of the analyzing ear to familiarize itself with the remote overtones."[2]

The goal of this chapter is the establishment of a pragmatic theory of tone representation, capable of accurately describing the aspect of our hearing which relates heard music to the harmonically referential extended just intervals. Such a theory should be applicable to a range of musical styles, though it is likely to be most effective in unfamiliar contexts. In musical styles that evoke strong stylistic and theoretical preconceptions in a listener, those preconceptions of style-specific rules are likely to overshadow the subtleties of tone representation; but when fewer culturally conditioned

[2] Schoenberg, op. cit., 21.

cues are available, tone representation emerges as one of the most effective ways of coming to terms with the unfamiliar.

In the interest of the greatest possible flexibility and transparency, I've presented my theory in the form of three simple preference rules; these are outlined in detail below. Judicious application of the preference rules allows the analyst considerable freedom in balancing different musical and contextual criteria, without losing sight of how particular choices privilege one aspect of the theory over another. For example, the theorist can choose different degrees of intonational tolerance for a just intonation string quartet and an equal-tempered piano piece, or confine tone representations to particular prime limits or harmonic spaces defined by certain integers. Keeping this degree of flexibility is a useful alternative to the wholesale adoption of Terhardt's algorithm or Parncutt's adaptation of its principles: despite the admirable predictive power of these theories, their computational complexity makes them inflexible and difficult to tailor to specific musical situations.

The most important antecedents for my theory are Hugo Riemann's *Tonvorstellung* and James Tenney's harmonic space; each of these theories is examined at length in the sections that follow. My own theory is outlined in the section "Tone Representation," which considers the theories of Riemann and Tenney and addresses some of the broad theoretical issues which they raise. After a brief review of the basic techniques for working with interval ratios, each of the preference rules is considered in turn, with emphasis on methods for practical application. At the end of the chapter, the theory is demonstrated in an analyses of an excerpt from Ligeti's *Melodien* and the "chorale" in Schoenberg's Piano Piece Op. 11/2.

Hugo Riemann and Tonvorstellung

In his idealist approach to music theory, Riemann asserts that we are not merely passive recipients of musical sounds, but active interpreters of those sounds into logical structures; the mental representation of musical relationships is more important than their actual manifestation as sound.[3] Every musical tone is "represented" or "imagined" as part of a justly tuned major or minor triad. (Riemann's work focuses on the primarily triadic music of the tonal tradition.) The harmonic meaning of each tone depends on its context—whether it is, for example, the third of a minor triad or the fifth of a major triad. A single, isolated tone may pose problems of ambiguity—but once we are familiar with the piece of music in which it occurs, it takes on a character depending on its harmonic representation: "According to whether a note is imagined as 1, 3, or 5 of a major chord or as I, III, or V of a minor chord, it is something essentially different and has an entirely different expressive value."[4]

Riemann labels the members of a major triad with the Arabic numerals 1, 3, and 5, and the members of a minor triad with the Roman numerals I, III, and V. In a minor triad, intervals are labeled from the fifth of the chord downward rather than from the root upward—this is in keeping with Riemann's dualist conception of the minor triad as an

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[3] Portions of the discussion of Riemann and Tenney in this chapter originally appeared in my 2006 article, "Tone Representation and Just Intervals in Contemporary Music," *Contemporary Music Review* 25/3 (June 2006): 263-281.

[4] Hugo Riemann, "Ideas for a Study 'On the Imagination of Tone'," translated by Robert Wason and Elizabeth West Marvin, *Journal of Music Theory* 36/1 (1992): 86. Ideas similar to *Tonvorstellung* are common in music psychology: the existence of mental templates corresponding to the diatonic scale has been proposed by Roger Shepard and Daniel Jordan—see "Auditory Illusions Demonstrating that Tones are Assimilated to an Internalized Musical Scale," *Science* 226/4680 (1984): 1333-1334. As Richard Parncutt points out, this is a "cultural" version of the influence of schemata or templates on tone perception, and closely parallels the pattern-matching approach to virtual pitch recognition advocated by Terhardt and others: see Parncutt, *Harmony: A Psychoacoustical Approach:* 37. Easley Blackwood has explored the possibility of forming "diatonic scales" in a variety of tunings which split the octave into various microtonal subdivisions, in *The Structure of Recognizable Diatonic Tunings* (Princeton, N.J.: Princeton University Press, 1985).

upside-down version of the major triad. All of these triadic relationships are governed by just intonation: thus, the equal-tempered fifths and thirds of chords played on a piano are "imagined" as pure, just intonation intervals in the listener's mind. Heard intervals are transformed into just intervals in the mind through the process of *Tonvorstellung,* which I translate as "tone representation" or "tone imagination."[5] *Tonvorstellung* also provides a harmonic meaning for each pitch, associating it with its harmonic root by a just interval.

Riemann proposes a general principle governing the way that our minds understand tones harmonically—we prefer the simplest possible interpretation consistent with the music. "This *Principle of the Greatest Possible Economy for the Musical Imagination* moves directly toward the rejection of more complicated structures, where other meanings suggest themselves that weigh less heavily on the powers of interpretation..."[6] Thus, given a collection of pitches, we will understand them as connected by the simplest possible just ratios, even when our ears are confronted by the complex and irrational intervals of equal temperament: "...our organ of hearing fortunately is so disposed that absolutely pure intonation is definitely not a matter of necessity for it."[7]

[5] The history and philosophical implications of the term "Vorstellung" are examined by Michael Kevin Mooney in the fourth chapter of his dissertation, "The 'Table of Relations' and Music Psychology in Hugo Riemann's Harmonic Theory" (Ph.D. Diss., Columbia University, 1996): 181-209. See also Robert Wason and Elizabeth West Marvin's introduction to their translation of Riemann's "Ideen zu einer 'Lehre von den Tonvorstellungen'," *Journal of Music Theory* 36/1 (Spring 1992): 69-79, Brian Hyer's "Reimag(in)ing Riemann," *Journal of Music Theory* 39/1 (Spring 1995): 101-38, and William Mickelsen's *Hugo Riemann's Theory of Harmony: A Study* (Lincoln, Nebraska: University of Nebraska Press, 1977).

[6] Riemann, op. cit., 88.

[7] Ibid., 99.

James Tenney: tolerance and harmonic space

For Riemann, the possibilities of tone representation end with the tonal relationships found within a triad: just intonation thirds and sixths, fourths, and fifths. James Tenney has proposed some similar theoretical ideas, but from a vastly different aesthetic position—while Riemann was aesthetically conservative, Tenney is a major creative figure in American experimental music. Like Ben Johnston and Harry Partch, he expands the concept of just interval to allow more complex integer ratios. However, unlike these more strict just intonation advocates, Tenney allows what he calls *tolerance*—"the idea that there is a certain finite region around a point on the pitch height axis within which some slight mistuning is possible without altering the harmonic identity of an interval"[8]—see the discussion of tolerance in Chapter 1, Premise 4. Tenney's idea that we hear intervals as "representing" the "simplest interval within the tolerance range" comes very close to Riemann's "Principle of the Greatest Possible Economy for the Musical Imagination," though Tenney is open to far more complex ratios as the basis of referential just relationships than Riemann's triadic possibilities.

Tenney separates the exact size of a heard interval from its harmonic sense; thus, we can imagine two different heard intervals representing the same harmonic sense, or the same heard interval representing two different harmonic senses. As an example, an equal temperament major third (400 cents) played on a piano can represent a just 5/4 major third of 386 cents. The equal-tempered third is audibly sharper than the smaller just third, but we can still identify it as projecting the "harmonic sense" of the just interval. One might object that there is a just ratio that much more closely approximates the equal

[8] Tenney, "The Several Dimensions of Pitch," 109.

temperament third: the 24/19 interval of 404 cents. However, due to Tenney's "simplest interval rule, we should tend to understand the 400 cent equal temperament third as 5/4, not 24/19.[9]

In a 1987 interview, Tenney defines "simplest" as equivalent to "the most compact arrangement in harmonic space":

> Given a set of pitches, we will interpret them in the simplest way possible. This can be translated into harmonic space terms by saying that it will be the most compact arrangement in harmonic space. Well, I think compactness, in that sense, could be measured somehow, and could be made very explicit by speaking of the sum of harmonic distances among these various points. So you could go through a piece and say, "Alright, we've heard in the beginning of the piece two pitches. You take the simplest ratio representation of that interval—tempered. Now we hear the third pitch. What specific, rational intonation for that approximate pitch will give us the simplest configuration in harmonic space, the most compact configuration in harmonic space? Let's call it that."[10]

The idea of harmonic space is defined in Tenney's 1984 essay "John Cage and the Theory of Harmony."[11] In harmonic space, each pitch is represented by a discrete point,

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[9] The context in which we hear an interval is very important in determining its harmonic sense: if we hear the equal-tempered third C-E as part of a larger set of pitches implying a fundamental of A, the context could lead us to understand it as the complex just ratio 24/19. If we hear the same third on its own, though, we are more likely to understand it as the simpler ratio 5/4, with a root of C.

[10] James Tenney and Brian Belet, "Interview with James Tenney," *Perspectives of New Music* 25/1-2 (1987): 462. Tenney's theory that we prefer simple explanations is closely related to the Gestalt psychology principle of *Prägnanz* (conciseness).

[11] Anyone familiar with Cage's biography will find this last title a strange juxtaposition—in a well-known anecdote, Cage's teacher Schoenberg warned him that his lack of a "feeling for harmony" would be "a wall through which [he] could not pass." Cage supposedly replied, "In that case, I shall devote my life to beating my head against that wall." Cage, *Silence,* 261. In "John Cage and the Theory of Harmony," Tenney proposes the need for a harmonic theory based on the psychology of harmonic perception. Such a theory would be descriptive, not prescriptive—it would not set rules for harmony in a particular style, but would instead describe the perceptual result of any harmonic choice. Tenney wants his theory to be as general and wide ranging as possible—pertinent to music of any time or place. Here the influence of Cage becomes apparent—particularly Cage's desire to embrace all sounds as potentially musical. In keeping with Cage's open attitude toward musical material, Tenney seeks a harmonic theory that can apply to any set of pitches. James Tenney, "John Cage and the Theory of Harmony," in *Soundings 13: The Music of James Tenney* (Santa Fe, New Mexico: Soundings Press, 1984): 55-83. Also in *Musicworks* 27 (1984), 13-17. Reprinted in *Writings about John Cage,* ed. Richard Kostelanetz (Ann Arbor, Michigan: University of Michigan Press, 1993): 136-61.

and is understood as a ratio relation to a central reference pitch, labeled 1/1. A simple harmonic space is the Riemannian *Tonnetz,* with axes representing multiples of 3 and 5. Harmonic space can be extended to many dimensions, though, by the inclusion of higher prime numbers: thus, intervals including 7 or its multiples create a new axis, as do intervals invoking 11, 13, or other primes. Distance between pitches is measured by the summation of steps along each axis: the axes are weighted so that a step along the 7 axis (for example) is “longer” than a step along the simpler 3 axis. The details of measurements in harmonic space are explored in the discussion of Preference Rule 2, below.

Tone representation

In the theory proposed here, “tone representation” is defined as the identification of one or more tones as partials in relation to a theoretical root or fundamental.[12] The complex of root and partials will be referred to as a tone representation, and can be concisely notated as a root followed by a list of partial numbers in ascending order. For example, if we understand an E above middle C as the fifth partial of a bass C2, we could describe that understanding with the tone representation C2(5). If we identify the fourth E4-A4 as a just perfect fourth, we can notate this tone representation as A2(3:4). (For consistency of notation, the partial numbers of intervals always are listed in ascending order, though this breaks the standard just intonation convention of notating them within

[12] The term “representation” must be used carefully; in the current context, it could be taken in several different senses. 1) We mentally “represent” heard pitches as points on a harmonic template based on the overtone series; 2) Our understanding of heard pitches is summarized in a “tone representation”; 3) the heard pitches are “representatives” of some underlying reality. The first two readings, which emphasize representation as a cognitive process are compatible with my theory; the third, with its strong tendency toward Platonism, seems philosophically untenable and should be avoided.

the octave 1/1 to 2/1.) The experience of a dominant seventh chord on G as comprising a major chord with an added natural seventh is shown in the tone representation G(4:5:6:7).[13] Partial numbers are particularly useful for the kind of harmonic analysis proposed here: they can easily be converted into actual frequencies by multiplying the partial number by the frequency of the fundamental, but are less cumbersome to work with and offer a kind of "movable-do" solfège in relation to the fundamental. For example, the partial number 17 always represents a pitch four octaves and (approximately) 105 cents above the root.[14]

The root itself can be notated with varying degrees of specificity: as a simple note name, as a note name plus/minus cents deviation (for microtonal pitches), and with or without an indication of register (e.g. C_4 = middle C). Whether or not register is included will depend on the musical situation: for example, if we are discussing representations of pitch classes instead of pitches in register, the register of the root is irrelevant. Where it is obvious or can be easily assumed from context, the root information can be omitted and taken as read: for example "the interval C4-F4 is represented as 3:4" implies a theoretical

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[13] A related notational convention can be found in Erkki Huovinen's *Pitch-Class Constellations: Studies in the Perception of Tonal Centricity* (Turku: Suomen Musiikkitieteellinen Seura, 2002). Huovinen, a music psychologist, studies how listeners choose tonal centers when presented with series of pitches. He develops the intriguing concept of the *pitch-class constellation:* "a tonally interpreted pitch-class set in which the subjective, local tonal center is taken as pitch-class 0." In his example of this notation, Huovinen compares two interpretations of the pitch set A-C-E-G. If it is understood as a seventh chord centered on A, it is notated A[3, 7, 10]; if it is understood as a C major triad with an added sixth, it is notated C[4, 7, 9] (Erkki Huovinen, "Two Arguments for the Mental Reality of Diatonicism: A Reply to Eytan Agmon," *Music Theory Spectrum* 28 (2006), 141-153: 144). Such a notational convention is clearly useful for music theory, a field in which questions like "Do you hear that as ^1 of G or ^5 of C?" are so common.

[14] This definition of tone representation assumes that the mapping between a given set and its representation is one-to-one. If we don't require that the mapping of the input set to the overtone series is one-to-one, cases could arise where notes that are close together (for instance, the adjacent quartertones in some works by Giacinto Scelsi) both map into the same tone representation—while not unthinkable, this complicates the calculation of tone representations discussed below. To avoid this situation, we could resolve any such "blurred" notes into a single pitch before subjecting the chord to tone representation analysis.

root of F2. Only when discussing abstract interval sizes without a specific pitch class context will we make tone representations without any root: "the equal temperament tritone is represented as 12:17."

This definition of tone representation and the associated notational convention strongly favor interpretations where all the pitches subjected to tone representation are overtones of the root: it does not make allowances for "subharmonic" relationships to a root. An example of a subharmonic relationship would be the hearing of the C4-F4 fourth with C as the root instead of F. Since F is not an overtone of C, this is an unlikely tone representation in my system: F cannot be represented as a whole number partial of the C root. Though I've found the "overtone only" approach adequate for the analyses presented here, it is possible to imagine musical situations where subharmonic tone representations would be desirable: e.g., taking the trichord C-F-G as centered on C, not F as my theory would imply with the tone representation F(6:8:9). One adaptation that would open the door to subharmonic tone representations is allowing fractions in the list of partial numbers: with this modification, we could notate the C-F-G trichord as C(2:8/3:3). The proportion between the numbers remains the same, but the "1" of the ratio is assigned to a different pitch.

An essential part of any theorizing involving tone representation is empirical testing. An effective way to test a root assignment is playing or singing that root along with the sonority in question—a convincing root assignment should fit with all the chord members without substantially changing how they are perceived in relation to one another. The root is most effective when in its correct registral position as "1" of the tone representation ratio. Similar *Hilfsvorstellungen* (auxiliary aids to imagination or representation) are the lower partials of the root, 3 and 5—these pitches should "fit" convincingly with the sonority if the tone representation is correct. In effect, by adding any of these pitches, we are examining their fit with empty notches in the "harmonic template" proposed by Terhardt's theory of virtual pitch. As in Terhardt's theory, there may be cases where two or more convincing interpretations make tone representation ambiguous; such ambiguity is not necessarily undesirable, and may accurately reflect multivalency in a work's structure.[15] I agree with David Lewin's defense of multiplicity when comparing his analysis of Stockhausen's *Klavierstück III* with Nicholas Cook's contrasting reading: "The differences in segmentation between Cook's analysis and mine should not be problematic, I think, except for those who believe that a form is 'a Form,' something a piece has one and only one of in all of its aspects."[16]

The theory of tone representation advanced here is in some sense the opposite of Milton Babbitt's view that atonal compositions are best understood as "contextual"—that is, based solely on structures introduced within the individual work. Instead, the ratio-based model that I discuss here is a theory of harmonic perception that can be applied across a broad repertoire of very different works. This is not to deny the importance of contextual relationships in these compositions, but to acknowledge that certain aspects of our harmonic understanding are likely to remain consistent from piece to piece.

These consistent aspects of harmonic perception are precisely those which are addressed by tone representation. The theory's main advantages include: 1) a clear way of describing our intuitions about rootedness and relative consonance and dissonance; 2)

[15] Bregman (op. cit., 232-239) describes how an inharmonic complex of partials will often have a "weak" global pitch: the pitch finding mechanism does the best it can to assign a pitch, balancing different implications of the complex.

[16] David Lewin, *Musical Form and Transformation* (New Haven, Conn.: Yale University Press, 1993): 62.

analytical sensitivity to register, which is often ignored by distance-based theories; and 3) the ability to make sense of complex harmonies that are difficult to engage with other approaches. These strengths make tone representation a valuable tool for pursuing an interopus understanding of harmony in many different styles of twentieth-century and contemporary music.

Working with interval ratios

Scholars of historical music theories and just intonation composers may be well acquainted with the procedures of working with interval ratios, but to most contemporary musicians such calculations are unfamiliar. I offer a brief summary of the most important techniques here. Readers already familiar with the basic principles of ratio intervals may wish to skip to the next section.[17]

notation: Any just interval *a*:*b* can be understood as the interval between partials of a harmonic series with the same numbers. Since Harry Partch, just-intonation composers have used ratios to define pitches, not just intervals. To use ratios to denote pitches instead of intervals, a given 1/1 must be chosen. If we assign 1/1 to a given pitch, 3/2 will describe the pitch a perfect fifth above, 6/5 the pitch a minor third above, and so on. In most current literature on just notation, pitch names are treated as pitch classes and represented as ratios in the single octave between 1/1 and 2/1. Intervals outside that range are multiplied or divided by 2 (equivalent to transposition of one of the pitches by an

[17] David B. Doty's *Just Intonation Primer* (San Francisco: The Just Intonation Network, 1993) is a very clear introductory text on these principles; see also Kyle Gann's website, "Just Intonation Explained" (http://www.kylegann.com/tuning.html, accessed April 15, 2008). An active online discussion group on tuning issues (especially in extended just intonation and non-twelve-tone temperaments) is located at http://launch.groups.yahoo.com/group/tuning/.

octave) until they fall within these bounds. For example, 9/4 (a major ninth above 1/1) would be changed to 9/8 (a major second above 1/1), and 15/32 (a minor ninth *below* 1/1) would be changed to 15/8 (a major seventh above 1/1).

As described above, I will notate tone representations as a root followed by a list of partial numbers in ascending order, separated by colons: for example, the tone representation F(5:7:9:11) describes four pitches heard as the 5th, 7th, 9th, and 11th partials of F. The root of a tone representation is equivalent to the pitch 1/1 in standard just intonation notation. When referring to just intervals outside the context of tone representation, I will use ratios without an root designation in the standard high/low order of appearance (3/2, 9/8, etc.).

addition: To add ratio intervals, multiply the ratios together. A major third (5/4) plus a minor third (6/5) = 5/4 × 6/5 = 30/20 = 3/2 = a perfect fifth.

subtraction: To subtract interval A from interval B, divide B by A. In ratio terms, this means multiplying A by the reciprocal of B. An octave (2/1) minus a major third (5/4) = 2/1 ÷ 5/4 = 2/1 × 4/5 = 8/5 = a minor sixth.

limits: Harry Partch introduced the useful idea of "limits." A just intonation system has a limit equal to the highest prime number used as a factor in its ratios. Thus Pythagorean just intonation, which uses only the primes 2 and 3 (and their multiples) in its ratios, is a three-limit system; Renaissance just intonation, which adds 5 and its multiples, is a five-limit system. Extended just intonation in Partch's formulation went to the eleven-limit, while Ben Johnston has invoked primes as high as 31 (though usually in a subsidiary role).

converting ratio intervals to cents: To change a ratio *a/b* into cents, we use the formula $c = 1200 \times \log_2 a/b$. Converting cents into a ratio is a much less common operation—this is because most cent values yield complex irrational numbers instead of simple ratios. The formula to find *a* given *b* and an interval *c* in cents is $a = b \times 2^{c/1200}$.

Figure 2.1 lists all of the intervals (within an octave) which use the partial classes 1 to 21. (The limitation to partial classes below 21 reflects the difficulty of accurately recognizing intervals beyond that limit, though arguably in certain circumstances much more complex intervals can be accurately perceived.) They are sorted in steps of one-sixth of a semitone; the symmetrical layout of the chart puts each interval opposite its inverse (for example, 14:17 is opposite 17:28). The size of each just interval is shown in cents as a subscript before its ratio. Many heard intervals have more than one possible tone representation: for example, the equal-temperament semitone can be closely approximated by the tone representations 18:19 (94 cents), 17:18 (99 cents), or 16:17 (105 cents).

$_{597}$17:24	$_{603}$12:17
$_{583}$5:7 $_{583}$15:21	$_{617}$7:10 $_{617}$21:30
$_{563}$13:18	$_{637}$9:13
$_{543}$19:26 $_{551}$8:11	$_{649}$11:16 $_{657}$13:19
$_{529}$14:19 $_{537}$11:15	$_{663}$15:22 $_{671}$19:28
$_{498}$3:4 $_{498}$9:12 $_{498}$15:20 $_{498}$21:28	$_{702}$2:3 $_{702}$6:9 $_{702}$10:15 $_{702}$14:21
$_{464}$13:17 $_{471}$16:21	$_{729}$21:32 $_{736}$17:26
$_{446}$17:22 $_{454}$10:13	$_{746}$13:20 $_{754}$11:17
$_{435}$7:9	$_{765}$9:14
$_{409}$15:19 $_{418}$11:14	$_{782}$7:11 $_{791}$19:30
$_{404}$19:24	$_{796}$12:19
$_{386}$4:5 $_{386}$12:15	$_{814}$5:8 $_{814}$15:24
$_{359}$13:16 $_{366}$17:21 $_{370}$21:26	$_{830}$13:21 $_{834}$21:34 $_{841}$8:13
$_{347}$9:11	$_{853}$11:18
$_{336}$14:17	$_{864}$17:28
$_{316}$5:6 $_{316}$15:18	$_{884}$3:5 $_{884}$9:15
$_{298}$16:19	$_{902}$19:32
$_{281}$17:20 $_{289}$11:13	$_{911}$13:22 $_{919}$10:17
$_{267}$6:7 $_{267}$18:21	$_{933}$7:12 $_{933}$21:36
$_{248}$13:15 $_{254}$19:22	$_{946}$11:19 $_{952}$15:26
$_{231}$7:8 $_{231}$21:24	$_{969}$4:7 $_{969}$12:21
$_{217}$15:17	$_{983}$17:30
$_{204}$8:9	$_{996}$9:16
$_{182}$9:10 $_{193}$17:19	$_{1007}$19:34 $_{1018}$5:9
$_{165}$10:11 $_{173}$19:21	$_{1027}$21:38 $_{1035}$11:20
$_{151}$11:12	$_{1049}$6:11
$_{128}$13:14 $_{139}$12:13	$_{1061}$13:24 $_{1072}$7:13
$_{112}$15:16 $_{119}$14:15	$_{1081}$15:28 $_{1088}$8:15
$_{94}$18:19 $_{99}$17:18 $_{105}$16:17	$_{1095}$17:32 $_{1101}$9:17 $_{1106}$19:36
$_{81}$21:22 $_{84}$20:21 $_{89}$19:20	$_{1111}$10:19 $_{1116}$21:40 $_{1119}$11:21

Figure 2.1: Table of intervals between partial classes 1 to 21

Preference rules for tone representation

In translating a collection of heard pitches to a referential just-intonation set, we are guided by what Riemann calls the "principle of the greatest possible economy for the musical imagination."[18] We choose the simplest just-intonation pitch set which matches the heard pitches, while minimizing the amount of mistuning between the heard pitches and their just intonation counterparts. Because many factors combine to determine the simplest tone representation, it is difficult to completely formalize the theory—what I propose instead is a simple model based on preference rules, which gives intuitively satisfying results.[19] In my view, the flexibility of this model is not a weakness, but rather one of its greatest strengths—the way we understand tones and their relations needs to be context-sensitive to allow for the interaction of other musical parameters with our harmonic perception. The three preference rules outlined here suggest the most likely ways to interpret any given harmony, while allowing the analyst to weigh the impact of contextual factors.[20]

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[18] Riemann, op. cit., 88. As noted above, a similar principle appears in the writings of the James Tenney.

[19] Preference rules make their first appearance in music theory in Fred Lerdahl and Ray Jackendoff's *A Generative Theory of Tonal Music* (Cambridge: MIT Press, 1983). The rules proposed here appear in a somewhat different form in my article "Tone Representation and Just Intervals in Contemporary Music," *Contemporary Music Review* 25/3 (2006): 263-281.

[20] The preference-rule approach shares a high degree of flexibility for the application and interaction of rules with Gestalt psychology's approach to visual scene analysis. Describing a Gestalt mechanism for parsing visual scenes, Bregman suggests that "it would be a good rule of thumb to prefer to group surfaces that were similar in appearance to one another" in color, texture, brightness, etc. This rule of thumb might not always yield a correct representation of the scene, but "if this principle were given a vote, along with a set of other rules of thumb, it is clear that it would contribute in a positive way to getting the right answer." Bregman, op. cit., 24.

Like the theory of tone representation advanced here, Ernst Terhardt's algorithm for finding virtual pitch (discussed in Chapter 1) frequently locates several possibilities for the virtual pitch of a given set of components. Terhardt invokes criteria similar to my three preference rules to choose between competing interpretations: "One may consider as being most significant that virtual-pitch value which a) is indicated by the integrating interval comprising the greatest number of near coincidences; b) corresponds to the smallest subharmonic number *m;* and c) can be obtained to the smallest integrating interval" (Terhardt, op. cit., 169). In their general effect, these guidelines correspond respectively to my preference rules 3, 2, and

1) Prefer interpretations in which the referential just intervals correspond as closely as possible to the actual intonation of the music—that is, tone representations which require the least retuning from the heard intervals to the referential just intervals.

2) Use the simplest possible interpretation of a pitch collection: the tone representation with the simplest just intervals between its members. (Simple intervals have low integers in their frequency ratios when reduced to lowest terms.) The presence of the fundamental (or one of its octave transpositions) tends to considerably strengthen the plausibility of a tone representation.

3) Use the smallest possible number of fundamentals; invoke multiple fundamentals only if they yield a significantly simpler interpretation than is possible with a single fundamental.

These rules provide a framework for determining the most likely ratio representations for any set of pitches, and correspond closely with Riemann's economy principle and Tenney's idea that we make sense of pitches in the most compact possible arrangement in harmonic space. The rules are also flexible enough to allow interpretation based on the musical context—in the analyses which follow, the analytical claims can and should be checked by ear. In the subsections which follow, each of these preference rules is explored in detail.

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1. While Terhardt's algorithm can form the basis for a theory of chord perception (see the work of Richard Parncutt, for example), I find the approach outlined here more intuitive to apply in analysis, if perhaps less strictly formulated from a quantitative, cognitive-science standpoint.

1. Prefer interpretations in which the referential just intervals correspond as closely as possible to the actual intonation of the music—that is, tone representations which require the least retuning from the heard intervals to the referential just intervals.

This preference rule (unlike the larger theoretical argument of this dissertation) deals purely with pitch *distance* instead of just interval—it applies our common-sense notion that the best tone representation must closely match the input set in absolute pitch. The degree of tolerance we accept between the input set and its tone representation cannot be definitively fixed—rather, it will vary depending on musical context. In music intended for performance in twelve-tone equal temperament, for example, we might have to accept retuning of as much as 50 cents: for example when the 11th harmonic (551 cents above the pitch class of the fundamental) is approximated as an equal-temperament tritone. The complexity of the intervals which can be conveyed to a listener is highly dependent on the precision of intonation: the more complex just intervals require great precision in tuning if they are not to be confused with simpler nearby intervals. The approach taken here offers a way to build a list of the possible tone representations of a given pitch set, ranked from closest to least close fit. After this list is created, the remaining preference rules can be invoked to select the best tone representation from the list—usually this involves a compromise between considerations of close fit (as described by this preference rule) and the simplicity of the intervals in the tone representations (as stated in Rule 2).

* * *

With the mathematical tools outlined in Clifton Callender's *Music Theory Online* article "Continuous Transformations," it is possible to quantify how much retuning is required to map a given pitch class set onto a target just-intonation pitch class set—we can calculate the Cartesian distance between the two sets.[21] Using basic calculus, we can find the transposition of the just intonation set which minimizes this distance: the transposition with the least total retuning. The distance metric allows us to compare how well the input set fits with different just-intonation representations, and choose the best fit—the process can even be automated as a computer program, assuring that no reasonable representations are accidentally overlooked.

To calculate the just intonation set which most closely matches a given input set, a computer is useful to compare the fit of many possible just intonation sets. The number of just intonation possibilities quickly becomes unwieldy for calculations done by hand: if we consider four-note sets made up of the 17 odd numbers (partial classes) from 1 to 33, there are 2380 different tetrachords. To complicate matters further, we will need to consider different mappings of the input set to each of the possible approximations. Since we will avoid voice-crossings in these mappings, we can limit our investigations by ordering both the input set and each of the possibilities from low to high, but we will still need to consider each of the four circular permutations for every possible representation, resulting in 2380 x 4, or 9520 possibilities for tetrachords using the odd partials from 1 to 33.

For computational purposes, the most effective way of finding the closest set is by calculating the minimum Cartesian distance between the input set and each target set at various transpositions: this technique is presented in Callender's article. Cartesian

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[21] Clifton Callender, "Continuous Transformations," *Music Theory Online* 10/3 (2004), 26-31. Cartesian distance is synonymous with Euclidean distance: both caluculate the shortest line conencting two points. An alternative to Cartesian distance is the "city-block" metric, which sums the distance between the points on each axis. City-block metrics show the distance one would have to travel between two points while traveling *only* in directions parallel to an axis, much as one would navigate city streets laid out in a grid. Cartesian distance, on the other hand, would calculate this distance "as the crow flies."

distance between the sets P and Q is determined by the equation $d(P,Q)=\sqrt{\sum(q_i-p_i)^2}$. To find the transposition of Q that is closest to P, we will explore all transpositions of Q by a variable x: $d(P,Q+x)=\sqrt{\sum(q_i-p_i+x)^2}$. We wish to find the value for x which produces the minimal distance between P (our input set) and Q (the target set transposed by x)—this is the point where the derivative of f(x) = 0.

The advantage of the Cartesian distance metric over the intuitively simpler "city-block" metric is that with Cartesian distance, this graph has a single minimum—that is, there is one and only one transposition that minimizes Cartesian distance between two sets.[22] When a city-block metric is used, the same minimum value can sometimes extend over a range of transpositions—the distance graph's minimum may have a flat bottom instead of a single point. Using Cartesian distance makes it possible to use calculus to determine a single low point by calculating the derivative of the distance function and finding what value of x makes the derivative equal to zero—this precision is ideal for computational purposes.

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[22] At the minimum Cartesian distance, the distances are minimized between all of the points—because each distance q_i-p_i is squared, if any one distance is substantially larger than another, it will disproportionately change the overall Cartesian distance. Faced with a number of transpositions of the target sonority where the sum of absolute displacements (city block measurement) is equal, the Cartesian measurement finds the transposition which minimizes the displacement of any given voice—thus it seeks the transposition of the set which is closest to the center of all the possibilities, thus minimizing all displacements. For example, given the input set 0 100 200 350 and a set of possible tone representations comprising the tetrachords in equal temperament, the closest two equal temperament tetrachords are 0123 and 0124. The minimum city block distance is 50 for each—but to minimize the Cartesian distance, in each case the reference tetrachord must be transposed up or down by 12.5 cents: 0123 goes to 12.5 112.5 212.5 312.5 and 0124 goes to 1187.5 87.5 187.5 337.5. So, the city block distance between the input set and either transposed set is 12.5 + 12.5 + 12.5 + 37.5, or 75 cents, even though there's a transposition with a smaller city block distance of 50 cents, with motion in just one voice. Taking the Cartesian distances, though, $\sqrt{0 + 0 + 0 + 50 *50} = 50$ while $\sqrt{12.5^2 + 12.5^2 + 12.5^2 + 37.5^2} = 43.30$, which is a smaller value. Minimizing Cartesian distance means moving each voice a small amount rather than one voice by a larger amount. (With the scaling factor introduced by Callender, 43.30 would be scaled to equal 50 by multiplying by $2/\sqrt{3}$.) This is the main difference between Cartesian distance and city-block distance, which simply sums the distances between each q_i and p_i. Viewed broadly, though, the results obtained by the two metrics are quite similar. For more on distance metrics, see Rachel Hall and Dmitri Tymoczko, "Poverty and polyphony: a connection between music and economics," in R. Sarhanghi, ed., *Bridges: Mathematical Connections in Art, Music, and Science* (Donostia, Spain, 2007).

This is the basic algorithm:

1) the input set is sorted from low to high;

2) the sorted input set is compared to a list of all the possible tone representations, in each of the possible circular permutations;

3) for each of these possibilities, the transposition level which requires the least retuning is found;

4) all possibilities are sorted based on their distance from the input set after being transposed to the transposition level from Step 3. The result is a list of the closest matches from the list of possibilities, each matched with a specific fundamental and a distance value.

To trace the flow of this process, I offer here a demonstration with a much more restricted set of possible tone representations. The same computational principles will apply to the much larger set of tone representations pursued later in this chapter. Instead of allowing tone representations involving the odd integers from 1 to 33 (resulting in 680 trichords and 2380 tetrachords), for demonstration purposes we'll use the severely constrained set 1, 3, 5, 7, which has just four trichordal subsets. These subsets can be expressed in both partial classes and pitch classes expressed in cents.[23] In Figure 2.2, each trichord is arranged in ascending order within a single octave.

partial class			*pitch class in cents (C = 0)*		
1	5	3	0	386	702
1	3	7	0	702	969
1	5	7	0	386	969
5	3	7	386	702	969

Figure 2.2: Partial classes and pitches for trichordal subsets of 1, 3, 5, 7

[23] The term "partial class" is introduced on page 32, by analogy to pitch class. Stated briefly, a partial class is the collection of all the octave transpositions of a given harmonic partial: this collection is identified by number of the lowest partial. Thus, the 5th, 10th, 20th, and 40th partials all belong to the partial class 5.

Using pitch class rather than pitches-in-register here simplifies the table enormously—we only need one value for each pitch class instead of a different value for each registral position. Information about actual register can be restored later, when we examine the list of best-fitting just intonation sets expressed as pitch classes.

Given an input set—let's try 0 200 400—we can begin to explore which of these tone representations offers the closest fit. We will want to compare the distance between the input set and each of the available tone representations. Since we haven't determined yet which pitch of the input set will map onto each pitch of the possible representations, we need to check each of the possible rotations of the representations. Each possible tone representation is listed from low to high—rotations 1 and 2 add 1200 cents (an octave) as necessary to keep the pitches of each in ascending order. The best transposition of Q is calculated by finding where the derivative of f(x) equals zero.

Recall that $d(P,Q+x)=\sqrt{\sum(q_i-p_i+x)^2}$. To dispense with the square root, we define f(x) as $d(P,Q+x)^2 = \sum(q_i-p_i+x)^2$. When comparing sets of three pitches,

$f(x) = (q_1-p_1+x)^2+(q_2-p_2+x)^2+(q_3-p_3+x)^2$

Expanding each element by the binomial formula $(a+b)^2 = a^2+2ab+b^2$,

$f(x) = (q_1-p_1)^2+2(q_1-p_1)x+x^2 + (q_2-p_2)^2+2(q_2-p_2)x+x^2 + (q_3-p_3)^2+2(q_3-p_3)x+x^2$

or

$f(x) = 3x^2+ 2((q_1-p_1) + (q_2-p_2) + (q_3-p_3))x + (q_1-p_1)^2 + (q_2-p_2)^2+ (q_3-p_3)^2$

The derivative of f(x) is

$d/dx\ f(x) = 6x+2((q_1-p_1)+(q_2-p_2)+(q_3-p_3))$

Setting this to zero and solving for x, we see that

$d/dx\ f(x) = 6x+2((q_1-p_1)+(q_2-p_2)+(q_3-p_3)) = 0$

$6x = -2(q_1-p_1)+(q_2-p_2)+(q_3-p_3))$

$x = -((q_1-p_1)+(q_2-p_2)+(q_3-p_3))/3$

On page 31, Callender notes that this can be generalized to sets of size i: the best transposition equals $-\sum(q_i-p_i)/i$. We can thus determine the best transposition for each possible tone representation, but another step is required—we now want to compare the distance between the input set and each of the target sets *at their best transposition.* Again, we'll use the formula $d(P,Q)=\sqrt{}(q_1+x\ -p_1)^2+(q_2+x\ -p_2)^2+(q_3+x\ -p_3)^2$. Following Callender, we'll adjust this value by multiplying it by the constant $\sqrt{3}/2$: this satisfies the intuitive sense that the Cartesian distance between, say, 012 and 013 is 1, instead of 81.65 as provided by the steps above. Scaling the distance value in this way suggests a more intuitive measure for the degree of retuning between the two sets.

Figure 2.3 shows the best transpositions and Cartesian distances between the input set 0 200 400 and the various possible target sets made up of the partial classes 1, 3, 5, and 7. The best match, with the scaled Cartesian distance 86.8, is found with the partial classes 3, 7, and 1 transposed down by 757 cents to yield the pitch classes (in cents) 1145, 212, and 443. The retunings required to map P to Q+*x* are small: -55, 12, and 43 cents. Due to the transposition, the pitch class of the fundamental or root of this set is 443. The next best match is far less convincing as a tone representation: the set can be mapped to partial classes 5, 3, and 7 with a scaled Cartesian distance of 160.4, but substantial retunings—as large as an equal-tempered semitone—are required.

	p_1	p_2	p_3					
input set(P)	0	200	400					
				best trans.	cent dist. at best trans.			scaled Cart. dist.
original(Q)	q_1	q_2	q_3	x	q_1+x-p_1	q_2+x-p_2	q_3+x-p_3	
1 5 3	0	386	702	-163	-163	23	139	263.9
1 3 7	0	702	969	-357	-357	145	212	538.6
1 5 7	0	386	969	-252	-252	-66	317	502.5
5 3 7	386	702	969	-486	-100	16	83	160.4
rotation 1								
5 3 1	386	702	1200	-563	-177	-61	237	369.9
3 7 1	702	969	1200	-757	-55	12	43	86.8
5 7 1	386	969	1200	-652	-266	117	148	399.4
3 7 5	702	969	1586	-886	-184	-117	300	454.2
rotation 2								
3 1 5	702	1200	1586	-963	-261	37	223	422.9
7 1 3	969	1200	1902	-1157	-188	-157	345	518.2
7 1 5	969	1200	1586	-1052	-83	-52	134	203.3
7 5 3	969	1586	1902	-1286	-317	100	216	485.5

Figure 2.3: Calculation of the best-fitting just-intonation match for the set 0 200 400.

This methodology can be applied to any table of possible tone representations: one could, for example, substitute equal-temperament pitch-class sets for my just-intonation partial-class sets to find how closely an input set fits each pitch-class set. As the input sets get larger, the task becomes more and more complex, particularly if we allow complex tone representations including high primes. The number of potential just intonation sets of any given cardinality is theoretically infinite, but we can choose a practical limit on the complexity of ratios we wish to allow in tone representations. In the research for this study, I have arbitrarily set this limit at 33, which yields ratios of more than adequate complexity to account for music tuned in twelve-tone equal temperament or quartertones. (Music with more precise tunings—such as the compositions of James Tenney or La Monte Young—would require a higher limit to describe distant just intonation relationships.) The limit 33 is appealing as it encompasses just over five octaves, and permits many complex tone representations without flooding the researcher with implausibly distant representations, as might happen with a higher limit (49, say, or 65). The intervals 32/31 and 33/32 are each approximately a quartertone.

Within the partial class limit of 33, there are 17 distinct pitch classes: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33. Taken two at a time, this results in 136 intervals; taken three at a time, there are 680 different trichordal subsets of the 17 pitch classes; taken four at a time, there are 2380 tetrachords. These numbers can be determined by the equation n!/k!(n-k)!, where n is the number of elements and k is the number of items selected at a time. By this equation there are 6188 pentachordal combinations and 12376 hexachordal combinations.[24]

Note that we are interested here in combinations, not *set classes.* The pitch intervals between the members of our 17-note set are not equal, so transpositional equivalence is not a possibility. Here we are concerned instead with creating a list of possible tone representations for our input pitch class set. The number of possible tone representations with the partial class limit of 33 quickly makes it impractical to keep all possibilities in mind for an intuitive selection. For this reason, the computational approach described above becomes indispensable for large sets. For smaller set classes like trichords, we can often find the best match "by hand," without resorting to computer calculations—for this purpose, the chart of all just intonation trichordal set classes made up of partial classes 1 to 21 in Figure 2.4 is a useful guide. The chart lists each set class in ascending order beginning with 0 (this notation will be familiar from twelve-tone pitch set class theory); the "prime form" of each set (the form with the smallest intervals at the beginning) is listed on the left side, with the inversions on the right. Symmetrical sets, of course, have

[24] The interested reader may wish to consult Julian Hook's "Why Are There Twenty-Nine Tetrachords? A Tutorial on Combinatorics and Enumeration in Music Theory" in *Music Theory Online* 13/4 (December 2007).

no distinct inverted form. Because of the different just intonation implications of prime and inverted forms of the set classes, I have not assumed inversional equivalence as in canonical set theory. The just intonation sets which correspond to each set class are listed as a set of three partial numbers: for example, the set class 0 2 4.5 can imply either partials 16, 18, and 21 or 17, 19, and 22. The symbol *x* indicates that no just intonation set drawn from partial classes 1 to 21 fits the given set class.

Once we've determined several just intonation sets which closely fit the input set, we can choose between them by the application of Preference Rule 2, selecting the simplest just intonation set. If no just intonation set fits the input set reasonably well, we can turn to Preference Rule 3 and describe the input set as the combination of just intonation sets on two or more fundamentals, as in Rameau's "dual generator" derivation of the minor triad in the *Démonstration du principe de l'harmonie*.

"Prime forms"				Inversions			
0	**0.5**	**1**	x	**SYMMETRICAL**			
0	**0.5**	**1.5**	20-21-22	**0**	**1**	**1.5**	19-20-21
0	**0.5**	**2**	x	**0**	**1.5**	**2**	x
0	**0.5**	**2.5**	x	**0**	**2**	**2.5**	18-20-21
0	**0.5**	**3**	20-21-24	**0**	**2.5**	**3**	x
0	**0.5**	**3.5**	x	**0**	**3**	**3.5**	17-20-21
0	**0.5**	**4**	x	**0**	**3.5**	**4**	x
0	**0.5**	**4.5**	20-21-26	**0**	**4**	**4.5**	16-20-21
0	**0.5**	**5**	x	**0**	**4.5**	**5**	x
0	**0.5**	**5.5**	20-21-28	**0**	**5**	**5.5**	15-20-21
0	**0.5**	**6**	x	**0**	**5.5**	**6**	x
0	**1**	**2**	15-16-17 16-17-18 17-18-19 18-19-20	**SYMMETRICAL**			
0	**1**	**2.5**	13-14-15 18-19-21 19-20-22 21-22-24	**0**	**1.5**	**2.5**	12-13-14 14-15-16 19-21-22
0	**1**	**3**	15-16-18 16-17-19 17-18-20	**0**	**2**	**3**	15-17-18 16-18-19 17-19-20
0	**1**	**3.5**	13-14-16 17-18-21 18-19-22	**0**	**2.5**	**3.5**	13-15-16 14-16-17 18-21-22
0	**1**	**4**	15-16-19 16-17-20 19-20-24 21-22-26	**0**	**3**	**4**	11-13-14 15-18-19 16-19-20
0	**1**	**4.5**	13-14-17 16-17-21 17-18-22	**0**	**3.5**	**4.5**	13-16-17 14-17-18 17-21-22
0	**1**	**5**	15-16-20 18-19-24 21-22-28	**0**	**4**	**5**	12-15-16 15-19-20 21-26-28
0	**1**	**5.5**	13-14-18 15-16-21 16-17-22 19-20-26	**0**	**4.5**	**5.5**	10-13-14 13-17-18 14-18-19 16-21-22
0	**1**	**6**	17-18-24	**0**	**5**	**6**	12-16-17
0	**1**	**6.5**	13-14-19 15-16-22 18-19-26 19-20-28 21-22-30	**SYMMETRICAL**			
0	**1.5**	**3**	10-11-12 11-12-13	**SYMMETRICAL**			
0	**1.5**	**3.5**	14-15-17	**0**	**2**	**3.5**	9-10-11 17-19-21
0	**1.5**	**4**	11-12-14 12-13-15 19-21-24	**0**	**2.5**	**4**	12-14-15 19-22-24 21-24-26
0	**1.5**	**4.5**	10-11-13 14-15-18	**0**	**3**	**4.5**	10-12-13 16-19-21 17-20-22
0	**1.5**	**5**	12-13-16	**0**	**3.5**	**5**	9-11-12
0	**1.5**	**5.5**	10-11-14 11-12-15 14-15-19 19-21-26	**0**	**4**	**5.5**	8-10-11 11-14-15 15-19-21 19-24-26
0	**1.5**	**6**	12-13-17	**0**	**4.5**	**6**	17-22-24
0	**1.5**	**6.5**	11-12-16 14-15-20 19-21-28	**0**	**5**	**6.5**	9-12-13 15-20-22 21-28-30
0	**2**	**4**	8-9-10 15-17-19	**SYMMETRICAL**			
0	**2**	**4.5**	16-18-21 17-19-22	**0**	**2.5**	**4.5**	7-8-9 13-15-17
0	**2**	**5**	9-10-12 15-17-20	**0**	**3**	**5**	15-18-20
0	**2**	**5.5**	8-9-11 15-17-21	**0**	**3.5**	**5.5**	13-16-18 14-17-19
0	**2**	**6**	17-19-24	**0**	**4**	**6**	12-15-17
0	**2**	**6.5**	9-10-13 15-17-22	**0**	**4.5**	**6.5**	7-9-10 13-17-19
0	**2**	**7**	8-9-12	**SYMMETRICAL**			
0	**2.5**	**5**	6-7-8 18-21-24 21-24-28	**SYMMETRICAL**			
0	**2.5**	**5.5**	13-15-18 14-16-19 19-22-26	**0**	**3**	**5.5**	5-6-7 11-13-15 15-18-21 16-19-22
0	**2.5**	**6**	12-14-17	**0**	**3.5**	**6**	17-21-24
0	**2.5**	**6.5**	7-8-10 13-15-19 18-21-26 19-22-28 21-24-30	**0**	**4**	**6.5**	11-14-16 15-19-22 19-24-28 21-26-30
0	**2.5**	**7**	6-7-9 14-16-21	**0**	**4.5**	**7**	10-13-15 14-18-21 16-21-24
0	**3**	**6**	17-20-24	**SYMMETRICAL**			
0	**3**	**6.5**	11-13-16 15-18-22	**0**	**3.5**	**6.5**	9-11-13 13-16-19 14-17-20
0	**3**	**7**	10-12-15 16-19-24	**0**	**4**	**7**	4-5-6 12-15-18
0	**3**	**7.5**	11-13-17 17-20-26	**SYMMETRICAL**			
0	**3.5**	**7**	14-17-21	**SYMMETRICAL**			
0	**3.5**	**7.5**	9-11-14 13-16-20 17-21-26	**0**	**4**	**7.5**	11-14-17 21-26-32
0	**4**	**8**	12-15-19	**SYMMETRICAL**			

Figure 2.4: Quartertone trichords with partial classes 1 to 21, including possible just intonation representations

2. Use the simplest possible interpretation of a pitch collection: the tone representation with the simplest just intervals between its members. (Simple intervals have low integers in their frequency ratios when reduced to lowest terms.) The presence of the fundamental (or one of its octave transpositions) tends to considerably strengthen the plausibility of a tone representation.

After applying Preference Rule 1, we can say with confidence which possible tone representations require the least retuning to match an input set. In choosing among these possibilities, the main criterion is the relative simplicity of each representation: we seem to intuitively prefer the representation consisting of the simplest interval ratios. This observation recurs in the work of several theorists: earlier in this chapter, we saw versions of this principle from Riemann ("principle of the greatest possible economy for the musical imagination") and Tenney (most compact arrangement in harmonic space). The subject of this section is the precise definition of "simple": how we define simplicity will affect how we determine the most economical representation of a given pitch set. We shall also consider the role of register in choosing between tone representations, and the powerful effect of hearing the fundamental (or one of its octaves) as part of the sounding set.

Different rankings of consonances appear throughout the history of music theory, beginning with the distinction between perfect, imperfect, and intermediate consonances in Johannes de Garlandia's *De mensurabili musica* (c. 1250). As discussed in Chapter 1, an important factor in the consonance or dissonance of an interval is the relationship between the partials of its pitches—the beating between nearby partials is the cause of dissonance. This explanation, introduced by Helmholtz in the late nineteenth century, is still widely accepted today.[25]

[25] Helmholtz, op. cit., 185ff.

More than a century before Helmholtz's ranking of intervals by their sensory concordance (degree of beating), the mathematician Leonhard Euler proposed a purely numerical method of determining "degrees of smoothness" *(gradus suavitatis)* for any ratio interval.[26] According to the usual dictates of tuning theory, Euler begins with an interval expressed as a mutually prime ratio—that is, with any common factors eliminated. The *gradus suavitatis* is equal to the sum of the prime factors of both parts of the ratio, minus one less than the total number of prime factors. As an example, take the just minor third 6/5. The sum of the prime factors 3, 2, and 5 is 10; subtracting 2 (one less than the number of factors) gives the *gradus suavitatis* 8. Figure 2.5 reproduces Euler's chart of the intervals in each degree of consonance from 2 to 10.[27]

II	1:2
III	1:3, 1:4
IV	1:6, 2:3, 1:8
V	1:5, 1:9, 1:12, 3:4, 1:16
VI	1:10, 2:5, 1:18, 2:9, 1:24, 3:8, 1:32
VII	1:7, 1:15, 3:5, 1:20, 4:5, 1:27, 1:36, 4:9, 1:48, 3:16, 1:64
VIII	1:14, 2:7, 1:30, 2:15, 3:10, 5:6, 1:40, 5:8, 1:54, 2:27, 1:72, 8:9, 1:96, 3:32, 1:128
IX	1:21, 3:7, 1:25, 1:28, 4:7, 1:45, 5:9, 1:60, 3:20, 4:15, 5:12, 1:80, 5:16, 1:81, 1:108, 4:27, 1:144, 9:16, 1:192, 3:64, 1:256
X	1:42, 3:14, 6:7, 1:50, 2:25, 1:56, 7:8, 1:90, 2:45, 5:18, 9:10, 1:120, 3:40, 5:24, 8:15, 1:160, 5:32, 1:162, 2:81, 1:216, 8:27, 1:288, 9:32, 1:384, 1:512

Figure 2.5: Euler's table of the *gradus suavitatis* of various intervals

Euler's formula leads to some odd results. The interval 1:9, three octaves plus a whole tone, has the relatively low GS of 5—this interval, always a dissonance in Western music, is two degrees "more consonant" than the major third 4:5. Euler's chart also seems to overemphasize the effect of octave transposition on the consonance of an interval—

[26] Leonhard Euler, *Tentamen novae theoriae musicae* (Saint Petersburg: Academy of Sciences, 1739). Euler's *gradus suavitatis* is discussed in Michael Kevin Mooney, "The 'Table of Relations' and Music Psychology in Hugo Riemann's Harmonic Theory," PhD diss., Columbia University, 1996.

[27] The degree 1 is assigned only to the unison, which Euler does not consider a consonance because it does not combine two different pitches: see Mooney, op. cit., 12.

each octave transposition changes to GS by one degree. Euler's gradations seem too broad to be practically useful: do we really hear a septuple octave (1:128), a whole tone, (8:9), and a minor third (5:6) as sharing the same degree of smoothness?[28]

Our criticism of discrepancies in Euler's table points out a certain circularity in any attempt to use purely numerical measures to reflect our sensation of relative consonance. Each numerical system is judged by its match with a preconception about relative consonance—whether based on the musical practice of a particular style and era or on psychological study (whether formal or informal) of relative consonance. For a formula of relative simplicity to be acceptable, it must match our established view of the relative simplicity of intervals—it would thus appear to provide no new information. The appeal of such a formula, though, comes when exploring intervallic worlds which extend beyond current common practice: ideally, we can hope that a numerical system which matches our intuitions about the known intervallic world will give comparable results for an extended interval palette—though this is by no means certain!

An essential requisite for a metric of harmonic simplicity is the ability to compare simplicity for sets of three or more pitches, not just single intervals. Euler's solution to this problem is to base the *gradus suavitatis* on the prime factorization of the least common multiple of the pitch set; this approach is based on the same considerations as his ranking of intervals, and is subject to similar problems.

One simple way of comparing the complexity of two tone representations is to choose the representation with lower partial numbers: given two tone representations of the set C_5:D_5:E_5:$F\#_5$ as F_0(24:27:30:34) or D_2(7:8:9:10), we can easily recognize the greater simplicity of the second representation by its lower partial numbers. In comparing

[28] See Mooney, op. cit., 10-21.

tone representations for the same set, this is equivalent to choosing the tone representation with a higher virtual pitch: D2 is higher than F0. (The use of virtual pitch as a guide to relative "consonance" of a pitch set is common among "spectralist" composers.[29]) As a general rule of thumb, this criterion is useful, but it ignores the question of factorability: in our comparison of sets, we should also seek the representation with the simplest just intervals between its members: this fits our intuition that the tone representation with partial classes 8:10:12:17 should be simpler than 7:9:11:17 despite the higher virtual pitch of the second list. A comparison of the interval content of each tetrachord is shown in Figure 2.6.

8:10:12:17	*7:9:11:17*
8:10 = 4:5	7:9
8:12 = 2:3	7:11
8:17	7:17
10:12 = 5:6	9:11
10:17	9:17
12:17	11:17

Figure 2.6

Several of the intervals of the tone representation on the left reduce to simple just thirds and fifths, while the intervals on the right are tend to be complex relations between higher prime numbers. A reasonable *ad hoc* method of comparing tone representations would be to combine the two criteria, favoring the interpretation with the highest virtual pitch and simplest intervals between its members: if these two criteria do not coincide, a judicious balancing of the two desiderata could suggest a clear preference in most situations.

The following two sections explore two recent solutions to the question of harmonic simplicity, proposed by Clarence Barlow and James Tenney, which produce more quantifiable results than the rules of thumb described above. While neither

[29] Joshua Fineberg, "Musical Examples," *Contemporary Music Review* 19/2 (2000): 124-128.

approach is a perfect solution, the ways that they address the question underscore the parameters that must be taken into account when comparing the relative simplicity of harmonies.[30]

Clarence Barlow's harmonicity function

In a 1987 article, Barlow compares the cognitive effect of simple just intervals with the gravitational pull of one physical body on another:

> it is undisputable that a given interval with a complex numerical relationship in the direct vicinity of another, more harmonic interval, falls into the pull of the stronger one, as it were. It thus operates as an approximation (for instance: an interval with the frequency relationship of 100:199 is only 0.7% smaller than an octave and is therefore heard as an octave); this "bending into place" is the subject of this text.[31]

The "bending into place" Barlow describes is clearly parallel to the notion of tone representation advanced here. Having identified the harmonic simplicity of an interval—its harmonicity, in his terminology—Barlow seeks a way of quantifying the simplicity of various rational intervals. For Barlow, harmonicity is determined not only by the absolute size of the numbers in an interval's ratio (when reduced to simplest terms), but also the divisibility of those numbers: their prime limit. For example, while the intervals 27/25 and 29/23 are quite similar in the size of their constituent integers, 27/25 is easier to comprehend because it can be broken down into simpler steps Both 27 and 25 are products of simpler primes, 5 and 3, while 29 and 23 admit no such simplification.

[30] One metric for relative consonance not considered in detail here is a comparison of the virtual fundamentals of the sonorities according to Terhardt's virtual pitch algorithm: a lower virtual pitch is equivalent to a less consonant collection. This metric is popular among spectral composers, and does not require the input sets to be in precise just intonation ratios. For the current purpose, though, a metric based on comparison of ratio intervals, such as Barlow's or Tenney's, is more useful.

[31] Clarence Barlow, "Two Essays on Theory," *Computer Music Journal* 11/1 (Spring, 1987): 44-60: 44.

Barlow proposes an "indigestibility" function, which gives a numerical value which reflects both the simplicity of an interval and the simplicity of its prime factors. Each integer can be described as the product of prime number factors: the indigestibility of any integer *n* is equal to the sum of $2(p-1)^2/p$ for each of the prime factors *p*. Barlow's list of the indigestibility of the integers 1 to 16 is listed in Figure 2.7.

n	*prime factors*	*indigestibility*
1	1	0
2	2	1
3	3	2.6667
4	2 × 2	2
5	5	6.4
6	3 × 2	3.6667
7	7	10.2857
8	2 × 2 × 2	3
9	3 × 3	5.3333
10	5 × 2	7.4
11	11	18.1818
12	3 × 2 × 2	4.6667
13	13	22.1538
14	7 × 2	11.2857
15	5 × 3	9.0667
16	2 × 2 × 2 × 2	4

Figure 2.7: Prime factorization and indigestibility of integers 1 to 16

Note how the indigestibility values for products conveniently add—for example, the indigestibility of 6, which we can notate as ind(6), equals ind(2) + ind(3). The indigestibility function leads to Barlow's harmonicity function for any interval expressed as a ratio in smallest (mutually prime) terms. For the ratio P/Q, "the more indigestible P and Q, the less harmonic the interval."[32] Harmonicity for the interval P/Q (with the higher integer as P) is defined by 1/ind(P)+ind(Q) when ind(P)-ind(Q) is positive and -1/ind(P)+ind(Q) when ind(P)-ind(Q) is negative. Negative harmonicities mean that the interval is "polarized to the higher note"—that is, that the higher integer in the ratio has is

[32] Barlow, op. cit., 45.

more digestible than the lower one. Thus the just major third 5/4, with a harmonicity of 0.1190 is polarized toward the lower note, 4; while the just minor sixth 8/5 is polarized toward the higher note, 8, with a harmonicity of -0.1064.

5/4	ind(5)=32/5, ind(4)=10/5	1/(42/5)	= 5/42	= 0.119048
8/5	ind(8)=15/5, ind(5)=32/5	-1/(47/5)	= -5/47	= -0.106382
4/3	ind(4)=6/3, ind(3)=8/3	-1/(14/3)	= -3/14	= -0.214286
3/2	ind(3)=8/3 ind(2)=3/3	1/(11/3)	= 3/11	= 0.27272727

Figure 2.8: Harmonicity values for some simple just intervals

The harmonicity value of the unison 1/1 is considered infinite, and the octave has a harmonicity of 1.00. Barlow suggests that setting a minimum harmonicity value is a useful and historically suggestive way of limiting pitch sets—for example, when he sets the minimum harmonicity to 0.06, the least harmonic interval allowed within an octave is the 81/64 Pythagorean major third (harmonicity 0.0600)—the most harmonic, of course, are the unison, octave, and perfect fifth (harmonicity 0.272727). As Barlow notes, the minimum harmonicity value of 0.1065 allows only the pitches of a just intonation diatonic scale. Intervals invoking the eleventh harmonic only appear when the minimum harmonicity is lowered to 0.04, while lowering the minimum harmonicity to 0.03 allows intervals with the thirteenth partial. For groups of three or more pitches, Barlow sums the absolute values of the harmonicities of all the possible intervals between constituent pitches.

By examining the "specific harmonicity" of sets, allowing a variable tuning tolerance so each pitch is movable (usually Barlow allows 20 cents), Barlow can search for the simplest tuning of a given shape (a scale, chord, etc.) by maximizing the specific harmonicity. This makes it possible, for example, to find the just tuning for a whole tone scale (accepting again 20 cents of possible retuning) which has the greatest overall harmonicity. Barlow's solution finds two equally harmonic tunings, each with a total harmonicity of 0.1882:[33]

1/1	9/8	5/4	45/32	8/5	9/5	2/1
1/1	10/9	5/4	64/45	8/5	16/9	2/1

The relevance of this procedure to my second preference rule should be clear—Barlow's sum of the harmonicities of all intervals in a set is a way of quantifying the relative simplicity of a given tone representation. This is one effective way of formalizing this aspect of the theory—in the next section, we will explore James Tenney's "harmonic distance" function as a potential alternative.[34]

James Tenney's harmonic distance

Barlow's computational approach to harmonicity yields results that seem in line with our harmonic intuitions, but it is difficult to visualize the abstract calculations that give rise to his harmonicity measures. James Tenney's approach to quantizing harmonic distance is more intuitive and far easier to visualize: it is based on a lattice model of just intonation pitch space, closely related to both the standard 5/3 Tonnetz and the extended just intonation lattices of Ben Johnston. In Tenney's pitch world, it is possible to add new dimensions to the lattice as necessary—each new dimension introduces a new prime factor.

Tenney's harmonic theory is based on three basic ideas (these basic ideas are closely related to my five premises in Chapter 1). The first is that intervals can be

[33] Ibid., 52.

[34] Richard Parncutt expresses doubts about Barlow's algorithm in "Applying Psychoacoustics in Composition: 'Harmonic' Progressions of 'Nonharmonic' Sonorities," *Perspectives of New Music* 32/2 (Summer 1994): 92. Parncutt argues that frequency ratios are less important for harmonic perception than pitch distances.

understood in two very different ways—either as up-and-down distances or harmonic qualities. The second is that these harmonic qualities can be matched to the just intervals—the intervals defined by simple frequency ratios. And the third is *tolerance*—the idea that we still recognize just intervals even when they aren't precisely in tune. Using these three basic ideas—the dual nature of interval as both distance and harmonic quality, the correspondence of interval qualities to just ratios, and tolerance for mistuned intervals—Tenney develops a multidimensional harmonic space, where each spatial dimension represents a different prime number factor. Figure 2.9 shows an example.

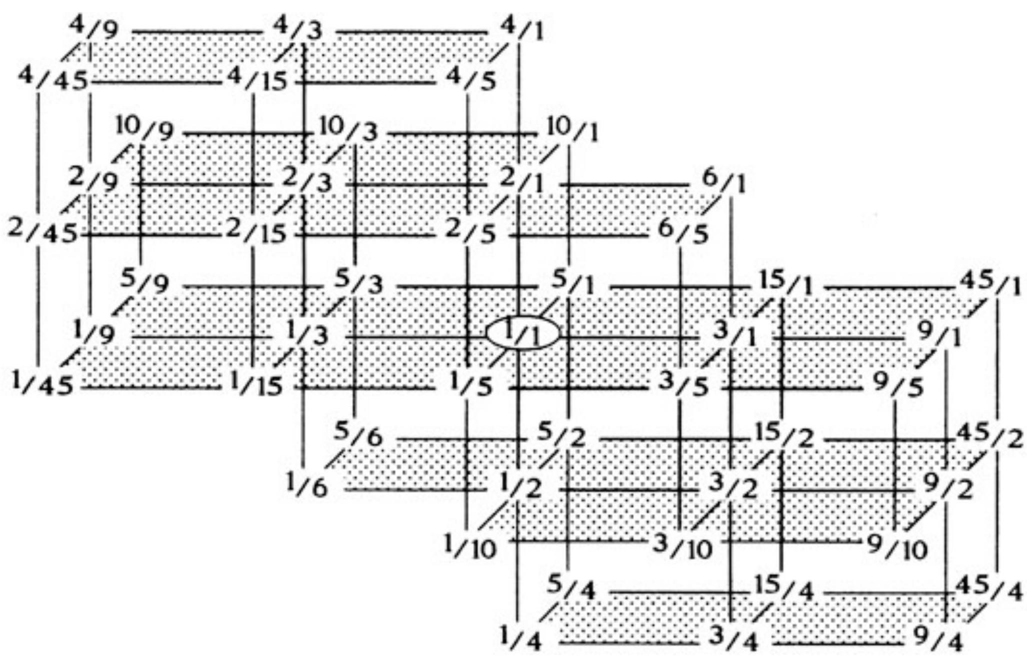

Figure 2.9: Tenney's harmonic space—in this case, limited to multiples of 2, 3, and 5[35]

This example is limited to just three dimensions, representing the factors 2, 3, and 5. Note that Tenney does not invoke octave equivalence, which would flatten this diagram into a familiar two-dimensional *Tonnetz.* Tenney's idea of harmonic space

[35] Example from James Tenney, "The Several Dimensions of Pitch," in *The Ratio Book,* ed. Clarence Barlow (Cologne: Feedback Studio Verlag, 1999).

allows the addition of more dimensions to account for higher prime factors—for example, a fourth dimension would be required to include the 7/4 natural seventh.

Tenney's harmonic space is discrete rather than continuous—tolerance means that pitches near a point on the lattice can be understood as representing that point. Roughly speaking, pitches close together in harmonic space are consonant with one another, while distant pitches are heard as more dissonant. Tenney provides a formula for calculating relative distances in harmonic space: given the ratio interval *a*/*b* reduced to its simplest (mutually prime) form, the harmonic distance between the pitches of the interval will be equal to the base two logarithm of *a* times *b*: $HD(Fa, Fb) = \log_2(a)+\log_2(b) = \log_2(ab)$. This assigns the value 1 to the smallest step in harmonic space, the octave. Other harmonic distances between ratios are shown in the table in Figure 2.10.

interval	1/1	10/9	9/8	8/7	7/6	6/5	5/4	9/7	21/16	4/3	11/8
cents	0	182	204	231	269	316	386	435	471	498	551
harmonic distance	0	6.49	6.17	5.80	5.39	4.90	4.32	5.98	8.39	3.58	6.46
interval	7/5	45/32	3/2	25/16	8/5	13/8	5/3	27/16	7/4	15/8	2/1
cents	583	590	702	773	814	841	884	906	969	1088	1200
harmonic distance	5.13	10.49	2.58	8.64	5.32	6.70	3.90	8.75	4.81	6.90	1

Figure 2.10: Distances for some simple just ratios according to Tenney's formula for harmonic distance.

The distance between any two points on the lattice is calculated by the sum of all the steps in between the points, but steps along the low prime-number axes are considered *shorter* than those along the axes of the higher primes. The axes are weighted by their logarithms base 2: thus, a step on the 2 axis is a harmonic distance of 1, a step on the 3 axis is a harmonic distance of 1.58, and so on. Steps along each axis can be summed for composite intervals: thus the perfect fifth 3/2 can be seen as a combination of one step on the 3 axis and one on the 2 axis, for a harmonic distance of 1 + 1.58 = 2.58. More complex intervals will require more steps—thus the harmonic distance between pitches in the ratio 15/4 is equal to the sum of the distances 3/1, 5/1, 1/2, and 1/2. A step along any of the prime number axes adds the base-2 logarithm of that prime to the harmonic distance of an interval.

Tenney's function provides a convincing metric for the relative harmonic distance between intervals. How can this be extended to compare the simplicity of sets with three or more pitches? Tenney proposes that "given a set of pitches, we will interpret them in the simplest way possible. This can be translated into harmonic space terms by saying that it will be the most compact arrangement in harmonic space."[36] This "compactness in harmonic space" is quantified in Tenney's article "On 'Crystal Growth' in Harmonic Space."[37] In this article, he explores how one might "grow" crystals in harmonic space by adding new points to a lattice one by one, each time at the point in the lattice which minimizes the *sum of all the harmonic distances* between points in the resultant set. This is comparable to Barlow's summation of all harmonicity values between pitches in a set to determine an overall harmonicity for the whole set: both Barlow's and Tenney's methods attempt to describe the simplicity or harmonicity of a set of three or more pitches by summing the simplicity values of all of the pairs within the set. Using Tenney's method of summing the harmonic distances between all the members of a set,

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[36] Tenney and Belet, "Interview with James Tenney," 462.

[37] James Tenney, "On 'Crystal Growth' in Harmonic Space (1993-1998)," *Contemporary Music Review* 27/1 (2008): 47-56.

we can compare the relative compactness or simplicity of multiple sets with the same number of members.[38]

Comparison of metrics for harmonic simplicity

Both Barlow and Tenney offer plausible ways of evaluating the relative harmonicity of just intonation pitch sets; we have also discussed how virtual pitch (after Terhardt) can be used as a measurement of harmonic simplicity. In this section, we will compare the efficacy of each metric in choosing the best tone representation for a specific pitch set: the chord in Figure 2.11, from Schoenberg's Piano Piece Op. 11, No. 2, m. 10.[39]

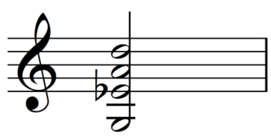

Figure 2.11: Chord from Schoenberg's Piano Piece, Op. 11, No. 2, m. 10

Figure 2.12 summarizes how the three metrics would rank the possible tone representations of the chord in Figure 2.11. The starting point for this comparison is a list of the 25 best-fitting tone representations, produced by applying Preference Rule 1. The leftmost column labels the sets from *a* to *y*. The degree of retuning required to map the sounding pitches to each candidate for tone representation is shown in the next column: the retuning distances range from 7.1 to 47.2 cents. The next column shows the fundamental of each tone representation, indicated in cents deviation from the nearest equal-temperament pitch class; the subsequent "8ve" column gives the register of the fundamental, in the standard notation where middle C = C4. The four central columns

[38] Only recently have attempts been made to treat larger sets holistically, not just as the sum of intervals: see Norman Cook and Takashi Fujisawa, "The Psychophysics of Harmony Perception: Harmony is a Three-Tone Phenomenon." *Empirical Musicology Review* 1/2 (2006): 106-126.

[39] This piece is discussed in more detail later in this chapter, with an analysis of the "chorale" in measures 9 to 13.

show the tone representations of each pitch in the chord. In the calculations of harmonicity and harmonic distance for each interval, the interval ratio between each two members of the tone representation would be reduced to its simplest (mutually prime) form. (This step is not shown here.) The three rightmost columns show the ranking of the candidates by each of the three metrics: Tenney's harmonic distance, Barlow's harmonicity, and the virtual fundamental (which follows the list of fundamentals given in column two: a high fundamental indicates a relatively simpler harmony). The list is sorted by the rankings produced by Tenney's metric, and the rankings of each tone representation by the other two metrics are given for comparison. In all cases, 1 is the simplest (most harmonic), and 25 is the most complex (least harmonic).

	retuning distance	root: *pitch class*	*8ve*	tone representations of: G	E-flat	A	D	Tenney ranking	Barlow ranking	Virtual pitch ranking
a	38.7	G$_{-12\text{¢}}$	*0*	8	13	18	24	*1*	*2*	3
b	42.0	E-flat$_{+19\text{¢}}$	*1*	5	8	11	15	*2*	*4*	1
c	7.1	C$_{-2\text{¢}}$	*0*	12	19	27	36	*3*	*3*	6
d	40.4	A$_{+27\text{¢}}$	*0*	7	11	16	21	*4*	*7*	2
e	29.5	G$_{+5\text{¢}}$	*-1*	16	25	36	48	*5*	*1*	10
f	38.7	E-flat$_{-1\text{¢}}$	*0*	10	16	23	30	*6*	*6*	5
g	34.7	F$_{+9\text{¢}}$	*0*	9	14	20	27	*7*	*5*	4
h	25.5	A$_{+41\text{¢}}$	*-1*	14	22	31	42	*8*	*11*	8
i	28.7	B$_{+12\text{¢}}$	*-2*	27	42	60	80	*9*	*9*	22
j	45.5	E-flat$_{+32\text{¢}}$	*-1*	20	31	44	60	*10*	*10*	15
k	47.2	D$_{+18\text{¢}}$	*-1*	21	33	48	64	*11*	*12*	17
l	27.8	E$_{+19\text{¢}}$	*-1*	19	30	42	56	*12*	*15*	14
m	26.3	B$_{+19\text{¢}}$	*-2*	25	40	56	76	*13*	*17*	21
n	35.5	F$_{-6\text{¢}}$	*-1*	18	29	40	54	*14*	*8*	13
o	31.8	D$_{+6\text{¢}}$	*-1*	21	34	48	64	*15*	*13*	18
p	33.2	A-flat$_{-6\text{¢}}$	*-1*	15	24	34	46	*16*	*18*	9
q	44.1	B-flat$_{+29\text{¢}}$	*-1*	13	21	30	40	*17*	*14*	7
r	36.0	C-sharp$_{-19\text{¢}}$	*-1*	23	36	52	68	*18*	*21*	20
s	27.1	A$_{-41\text{¢}}$	*-2*	29	46	66	88	*19*	*16*	24
t	45.7	C-sharp$_{-2\text{¢}}$	*-1*	23	36	50	68	*20*	*19*	19
u	31.1	F-sharp$_{+5\text{¢}}$	*-1*	17	27	38	50	*21*	*20*	11
v	38.5	F-sharp$_{-12\text{¢}}$	*-1*	17	27	38	52	*22*	*22*	12
w	40.8	A-flat$_{-20\text{¢}}$	*-2*	31	48	68	92	*23*	*25*	25
x	35.4	E-flat$_{-49\text{¢}}$	*-1*	21	33	46	62	*24*	*24*	16
y	42.1	A$_{-27\text{¢}}$	*-2*	29	46	64	88	*25*	*23*	23

Figure 2.12: Comparison of Tenney, Barlow, and virtual pitch rankings of harmonic simplicity

The three metrics give roughly similar results, though the Tenney and Barlow rankings resemble one another more than the virtual fundamental ranking. This is because, as noted above, the virtual fundamental is based solely on the lowest partial number of the tone representation, and otherwise disregards the complexity of the intervals within the set. This is why the virtual fundamental ranking finds tone representation *e* much less harmonic (at rank 10) than either the Tenney or Barlow metrics (ranks 5 and 1, respectively). The inclusion of the partial 25 makes all the partial numbers high in the overtone series (harmonic partials 16, 25, 36, and 48), resulting in a low virtual fundamental at G$_{+5\text{¢}}$-1. Used as a metric, the virtual fundamental is insensitive to the simple relationships in the proportion 16:36:48, which reduces to 4:9:12. This simple proportion accounts for the high ranking of this tone representation by the Tenney and Barlow metrics, despite the inclusion of the more distant partial number 25.

The Tenney and Barlow rankings identify the same simplest seven sets (ranks 1 to 7) and least simple four sets (22 to 25). The differences in the way they order the sets tend to reflect how much each method weighs the effect of more complex partial numbers in reducing a tone representation's degree of simplicity. Barlow's method tends to prefer stringing together lower primes to invoking higher primes: we see this in its top-ranked choice, which represents the pitch E-flat as 25 (5 × 5) for the same pitch Tenney's algorithm would designate as 13 (in *its* top-ranked choice). In general, Barlow's method gives a greater weight to interval simplicity: the presence of an interval made up only of multiples of 3 and 2 (three-limit intervals, in Harry Partch's terminology) contributes significantly (perhaps too significantly) to Barlow's overall harmonicity score. These simple ratios also contribute to shorter harmonic distances in Tenney's harmonic distance calculations: but the curve drops less sharply for Tenney, meaning that the inclusion of higher primes has a less drastic effect on the overall simplicity. Thus, it could be said that Tenney's metric is more liberal about the move to higher prime limits.

In making our selection from this list, any of the three metrics offer a reasonable way of reducing all the candidates to a few possibilities: we could use any of the metric to find a "top ten" of likely tone representations. At that point, we will need to consider additional factors besides simplicity to make a final choice. The first of these factors is the presence or absence of the fundamental: as the final clause of Preference Rule 2 states, "The presence of the fundamental (or one of its octave transpositions) tends to considerably strengthen the plausibility of a tone representation." Among the top contenders, this preference would favor *a, b, d, e,* and *f.* The presence of the fundamental as one of the *lower* members of the chord seems to have a greater impact than when it appears at the top of the chord: thus *a* and *e* seem particularly strong, since they identify the lowest sounding pitch as an octave of the fundamental (partial number 8 and 16 respectively). Tone representation *b* is also strong by this criterion, with the "tenor" E-flat as partial number 8. A second consideration is the balance of harmonic simplicity against Preference Rule 1's reduction of retuning from the heard set to the tone representation. In this case, tone representation *c* is particularly strong: it takes far less adjustment to fit the equal-tempered pitches of the heard chord to this just tone representation.

Given these considerations, I would select *a, b,* and *c* as the three top candidates for this chord's tone representation: in addition to scoring high in each simplicity metric, *a* and *b* contain an octave equivalent of the fundamental in the low register, and *c* convincingly minimizes retuning. I find Tenney's number 1 (*a*) more convincing than Barlow's number 1 (*e*)—to me, the representation of E-flat as 13 is more convincing than the representation as 25, particularly because it keeps the virtual fundamental from becoming too low. All of these decisions are highly contextual, though: in a musical framework that emphasized composite five-limit intervals like the augmented fifth C-G-sharp and avoided extended just relationships with primes like 13, Barlow's favored tone representation might seem more convincing than Tenney's candidate.

3. Use the smallest possible number of fundamentals; invoke multiple fundamentals only if they yield a significantly simpler interpretation than is possible with a single fundamental.

The third preference rule governs the segmentation of large sets into combinations of tones on different fundamentals. As noticed in the fifth premise of Chapter 1, our sense of harmonicity often leads to the division of complex pitch sets into multiple simpler harmonic subgroupings—the workings of this perceptual segmentation seem closely related to problems in visual scene analysis and Gestalt psychology, where proximity and similarity between objects play a major role.[40]

Like the first two rules, this rule follows Riemann's "principle of the greatest possible economy for the musical imagination": we minimize the number of different harmonic roots that are simultaneously active. We resolve pitches into a single harmonic root when we can, but when we can't, we find the simplest possible consistent explanation, invoking as few roots as possible. More than the other rules, though, this rule is highly context-dependent: segmentations will vary substantially based on stylistic assumptions as well as the memory of previous events within a work. The establishment in a work of a "prime limit" (after Partch) may have a strong effect on the complexity of the tone representations we are willing to entertain before supposing a new, separate harmonic entity.

Because of the sensitivity of this rule to context, the general discussion here will be brief. Bregman's "old-plus-new" heuristic will often be useful: stated succinctly, this proposes that "if part of a present sound can be interpreted as being a continuation of an

[40] The division of complex sonorities into simpler harmonic sonorities has appeared in a variety of music theories, including Schoenberg's segmentation of a chord from *Erwartung* discussed in Chapter 1. Rameau's "dual generator" approach to the minor triad in the *Démonstration du principe de l'harmonie* (Christensen, op. cit., 162-168) is a well-known example; the minor chord A-C-E has two generators: A-E yields the perfect fifth, and C-E the major third.

earlier sound, then it should be."[41] The division of complex chords into smaller subsets is demonstrated at length in Chapter 4, on Grisey's *Vortex Temporum.* In the remainder of this chapter, the application of the three preference rules for tone representation will be demonstrated in analyses of excerpts from Ligeti's *Melodien* and Schoenberg's *Piano Piece* op. 11, no. 2.

Tone representation in Ligeti's Melodien *(1971)*

Microtonality and overtone-based harmony fascinated György Ligeti during several periods in his long compositional career since as early as the 1950s: Richard Toop discusses an abandoned electronic piece from the late 1950s based on a "synthetically produced overtone series" of sine tones.[42] Toop quotes Ligeti on this abandoned work: "My idea was that a sufficient number of overtones without the fundamental would, as a result of their combined acoustic effect, sound the fundamental... I imagined that slowly, different composite sounds would emerge and slowly fade away again like shadows, I intended to produce forty-eight layers of sound."

Of another late 50s work, *Apparitions* (1959), Ligeti stated "I should like to add, apropos of *Apparitions,* that I have used the individual voices in the orchestra, especially the strings, as though they were partials—they are sounds in themselves—as though they were partials of an even more complex sound. Here there is an analogy to the work in the electronic studio."[43] This suggests a close connection between the proto-spectralist idea of scoring partials for an instrumental ensemble and the early cluster-harmony works

[41] Bregman, op. cit., 261.

[42] Richard Toop, *György Ligeti* (London: Phaidon, 1999): 60.

[43] György Ligeti, *Ligeti in Conversation* (London: Eulenburg, 1983): 90.

which first brought Ligeti widespread attention in the West. Whole-tone cluster formations can easily be heard as consecutive partials in the range from the seventh to the twelfth: this suggests a more complex and subtle harmonic meaning for such clusters than usually recognized by analysts who treat them as atonal sets.

We can thus see a thread of continuity in Ligeti's musical concerns from the beginning of his career to recent works like the *Hamburgisches Konzert* and Violin Concerto: this suggests new ways of thinking about harmony in works from *Apparitions* and *Atmospheres* to the 1965 *Requiem* that would take account of such "spectral" characteristics.

Ligeti's first period of exploring microtones in instrumental music begins in 1966, with the Concerto for Cello and Orchestra, and stretches into the early 1970s. During this period, Ligeti seems to be experimenting with various ways of using microtonal pitches. In the first movement of the Cello Concerto, Ligeti writes very high natural harmonics for the soloist—these include the microtonal 11^{th} and 13^{th} harmonics, which fall "between the keys" of the tempered twelve-note scale. In the String Quartet No. 2 (1968), explicitly notated microtones are used in cluster-like formations. *Ramifications,* with its two string ensembles tuned approximately a quartertone apart, turns microtonality into a formal principle—what's more, the interest in *scordatura* hints at developments to come in the 1990s. (Near the end of *Ramifications,* a *sul ponticello* double bass plays a timbre rich in high partials—these suggest the same kind of "between the keys" harmonics heard in the Cello Concerto, and reflect the texture of *scordatura* microtones in the rest of the piece.) The Chamber Concerto (1970) and *Melodien* (1971) are not microtonal works (they're written in standard twelve-tone notation), but bear the marks of Ligeti's microtonal experiments—here, overtone configurations are powerful shaping forces on the harmony at important junctures (see the analysis of an excerpt from *Melodien* below). In Ligeti's next work, the Double Concerto (1972), he has returned to explicitly notated microtones—the musicologist Bob Gilmore has suggested that Ligeti's 1972 encounter with the American microtonalist Harry Partch was a significant influence on this work.[44]

The mid and late 70s show Ligeti's attention turning away from these specifically microtonal issues, as he focused on the opera *Le Grand Macabre* and minimalist-inspired works such as *Monument-Selbstportrait-Bewegung.* Ligeti's interest in microtones emerges again in the early 1980s, with the Trio for Horn, Violin, and Piano (1982). (Ligeti's student Manfred Stahnke, a pupil of Ben Johnston, may have been a spur to new intonational experiments.) Here we see the use of just intonation achieved by the use of the natural harmonics of the horn. Such natural horn microtones are also prevalent in the Piano Concerto (1988) and throughout the *Hamburgisches Konzert* (1999/2002), for solo horn and orchestra with four natural horns. (The *Hamburgisches Konzert* also represents Ligeti's response to the spectral music of Grisey and Murail—see particularly the fifth movement, titled "Spectra.") The use of scordatura, as well as notated pitch inflections, adds microtonal color to the 1993 Violin Concerto, and as we have seen, the first movement of the Viola Sonata (1991-94) uses the overtone series as the basis for a mournful Romanian *hora lungă*.

The range of microtonal techniques found in Ligeti's work is remarkable—microtonality is a compositional issue that Ligeti returned to over and over again, with different solutions each time. Unlike many composers, Ligeti has never formalized his harmonic practice into any sort of strict system—rather he prefers to experiment with

[44] Bob Gilmore, "The Climate Since Harry Partch," *Contemporary Music Review* 22/1-2 (2003): 27-30.

various kinds of "hybrid microtonality," reinvented for each piece.[45] (Throughout his career, Ligeti tended to be suspicious of systematization, a tendency that put him at odds with the integral serialists of the 1950s and 60s.) In his pitch designs, Ligeti uses a mixture of intuition and tools from various theories, including just intonation. Tone representation is a useful tool in exploring these mixed harmonic approaches. Even with no single common compositional technique or organizing principle, the way the listener experiences these harmonies can be addressed by an analytical approach with strong roots in perception. Tone representation provides a neutral ground from which comparisons can be made between different eras of Ligeti's harmonic practice, or even between different harmonic worlds in the same piece.

* * *

Ligeti's *Melodien* for orchestra (1971) is *not* a microtonal piece (it's written entirely in standard twelve-tone notation), but certain passages strongly imply tone representations which involve microtonal intervals such as 8/7 (231 cents) and 11/8 (551 cents).[46] These just microtonal intervals are approximated to the nearest semitone—in a sense, then, one could argue that these passages are examples of just-intonation microtonal music forced into a semitone grid. If, as I would argue, we tend to bring the same harmonic intuitions to all kinds of music, the harmonic processes we've seen in just intonation contexts should also apply to music written in twelve-tone equal temperament. By recourse to the tools of tone representation, we can begin to see how an "atonal" work

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[45] The phrase *"hybrid mikrotonalitas"* appears in one of Ligeti's sketches for the Violin Concerto at the Paul Sacher Stiftung (Folder 1, 80).

[46] Portions of this analysis were originally published in my article "Tone Representation and Just Intervals in Contemporary Music," *Contemporary Music Review,* Vol 25/3 (June 2006): 263-281.

like *Melodien* might be understood instead as what Schoenberg called "pantonal"—that is, based on pitch relationships drawn from the higher reaches of the overtone series.

My analysis of the passage from mm. 11-19 describes the pitch structure as representing the upper partials of a shifting fundamental, invoking tone representations of partial classes up to 19. In the discussion that follows the analysis, I will compare my interpretation of the piece with Jonathan Bernard's analysis of the same passage as a series of distance-based transformations in pitch space. Bernard's analysis is efficient and convincing at describing distance-based relationships between the pitch sets of the passage, but fails (as it must, given its theoretical assumptions) to address the aspects of harmonic quality, rootedness, and tone representation which can be examined by ratio-based theoretical tools.

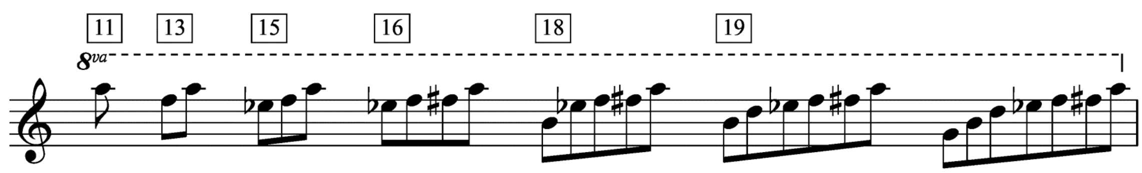


Figure 2.13: Pitch collections in Melodien, mm. 11-19

Figure 2.13 is an abstraction of all the pitches from m. 11 to m. 19 of *Melodien* (an excerpt of about 40 seconds). Bernard has discussed part of this section in terms of transformations in pitch space (that is, the space of pitches in register, as opposed to modular pitch class space).[47] In this approach, interval is conceptualized as distance; Bernard's analytical diagram of the music on graph paper, with each square representing a semitone, makes the analogy of interval to spatial distance explicit (see Figure 2.14).[48]

[47] Jonathan Bernard, "Ligeti's Restoration of Interval and Its Significance for His Later Works," *Music Theory Spectrum* 21/1 (Spring 1999), 1-31: 3-10.

[48] Note that Bernard's method here is even more purely distance-based than pitch-class-set theory because he excludes octave equivalence.

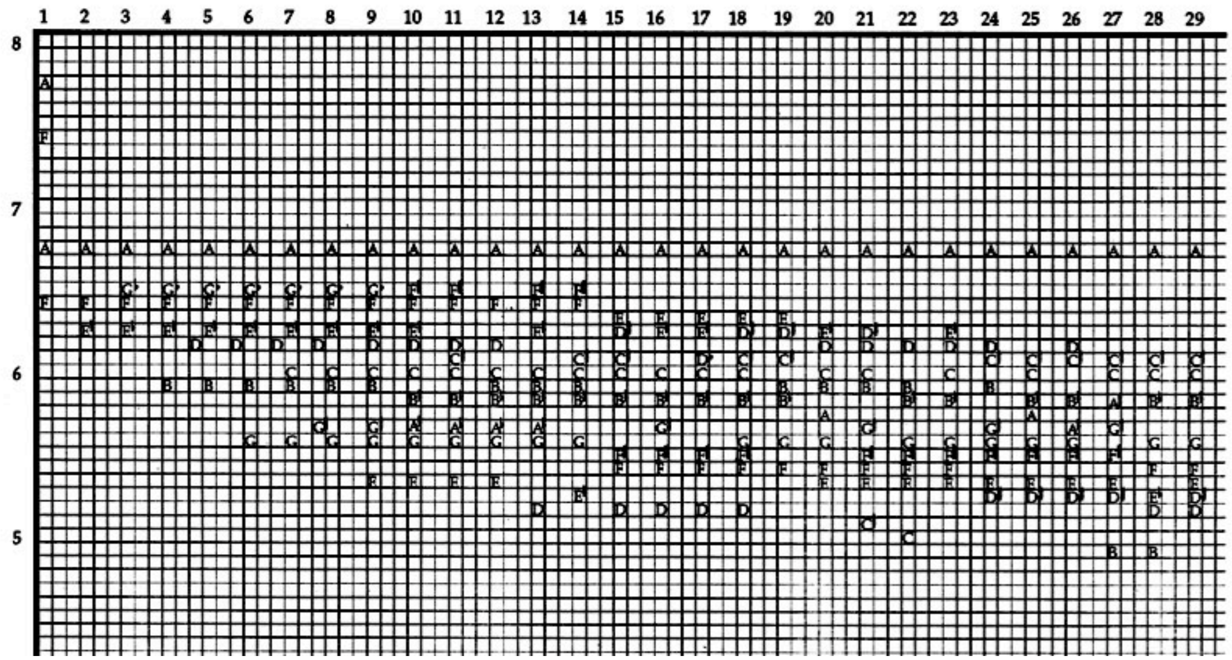


Figure 2.14: Bernard's graph "in pitch space" beginning at m. 13 of *Melodien.* The numbers 1 to 57 represent labels Bernard assigns to each of the individual "melodies" in this section.

In the passage of *Melodien* illustrated in his pitch-space graph, Bernard proposes a series of distance-preserving transformations in pitch-space to account for the progression from one "melody" to the next (Bernard, 15-18). These three transformations are:

1) the "flip," which moves a pitch from a distance X above a given pivot to the same distance X below (or vice versa);

2) the "spin," which inverts a set of three or more pitches in pitch space, with the axis of inversion chosen to keep them within the same pitch-space boundaries; and

3) the "glide," which replicates the distances between two or more pitches with added pitches at a different transposition level (or as Bernard prefers to conceive of the transformation, the pitches "are transposed to a new location without canceling their presence at the original location").

These transformations allow Bernard to make some powerful observations about the progression from one set of pitches-in-register to the next:

> Even a cursory glance at this graph reveals that only one note, the highest (A6), once established is retained by every subsequent melody in the passage. In other words, there is no fixed skeleton, but a highly mobile one, one that grows mainly out of the vertically adjacent interval content of the initial short melodies, gradually expanding and moving downward in range. The melodies still conform to these intervals, but in a somewhat unpredictable way: it is as if their sizes and qualities were fixed, rather than their positioning relative to one another. In fact, using a few simple operations it is possible to describe all of these successions of interval stacks—after the first few, that is, when a viable repertoire of intervals is still being assembled—as transformations in pitch space that deal directly with these interval qualities rather than with pitch-class or even pitch content.[49]

The problem with this mapping of pitch to an abstract spatial continuum (with interval as distance between points on that continuum) is that the map and the territory can easily be confused; it can become unclear whether the analysis is referring to the actual sound of the music, or merely describing patterns in the analytical graph. While Bernard does mention "interval quality," in this context, he seems to mean nothing other than "interval size." In the analyses of other Ligeti works that follow in this article, it seems clear that Bernard is not interested in engaging "quality" in the sonic sense which I discuss here—rather, he holds to the "distance" approach to interval.

One might of course argue that by constantly keeping in mind which pitches the points on the graph represent, the risk of getting lost in abstractions based solely on the graph is eliminated. While the importance of constant comparison of the phenomenon with its abstract model is essential for all analysis, in this case the limitations of interval-as-distance mean that the shapes and transformations described by Bernard renounce any essential connection with the sounding pitches. The scale of the graph is entirely

[49] Bernard, op. cit., 10.

irrelevant to Bernard's transformations, which could easily refer to a graph where each square represented an octave or a single cent. While the transformations would still hold true as a description of the new stretched or compressed music, the failure of the theory to discriminate between these scales seems to indicate a distressing lack of specificity.

Turning from distance intervals to just intervals, we can arrive at a very different reading of the passage as an example of changing tone representations. The excerpt begins with a unison A6. Given the lack of a context which would imply otherwise, it seems reasonable to assign this pitch the simplest possible tone representation of "fundamental": A(1)—see Figure 2.15.

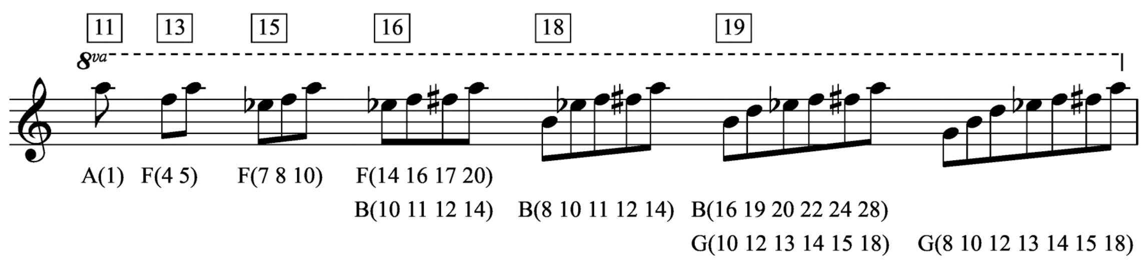


Figure 2.15: Tone representations in *Melodien*, mm. 11-19

At 13, an F appears below the A (this dyad is doubled an octave above by the celesta, but this does not significantly affect our harmonic understanding). The simplest, most "economical" way to hear the F/A dyad is as a just major third, F(4:5). At this moment, the tone representation of the A changes—it changes from a fundamental (1) to a just major third (5) above the new fundamental F (4).[50]

The addition of E-flat in m. 14 does not affect our sense of F as root—rather, it sounds like the approximated seventh harmonic of F(7:8:10). Other tone representations

[50] The constant reevaluation of different tone representations to select the simplest representation reflecting new data is an example of Daniel Dennett's "multiple drafts" theory of consciousness. See Dennett, *Consciousness Explained* (New York: Little, Brown and Co., 1991). Such a theory is comparable to Tenney's description of changing perceptions quoted on p. 93, above.

of these three pitches are possible, such as E-flat(8:9:11), B(10:11:14), G(13:14:18), or D(17:19:24), but with no reason to favor these more complex representations, the more economical F(7:8:10) is a better choice. The F has also been already established as a fundamental in measure 13, so a sort of "inertia" makes us likely to keep the same partial classes for the F and A. In music which rounds off microtonal just intervals to a semitone grid, it is necessary to accept larger degrees of tolerance than in music rounded to a quartertone grid—thus, the 200 cent interval E-flat/F is taken to represent 7:8 (231 cents) here, with a 31 cent discrepancy between the heard and referential intervals. While F and A are still represented by the same partial classes as in m. 13, the representation of the sonority must be understood as occurring an octave higher in the harmonic series (changing the representation of F from 4 to 8), to allow the inclusion of E-flat as F(7). The move into higher integers corresponds with the addition of more complex and less traditionally consonant intervals.

At measure 16, the addition of an F-sharp creates a harmonically ambivalent situation. We can fit the new collection into a F spectrum, F(14:16:17:20), which requires a move still higher into the overtone series, reinterpreting F from 8 to 16. However, a competing interpretation emerges which better satisfies the "Law of Economy": we can hear the collection as an approximation of B(10:11:12:14). It's difficult to choose between the two; our sense of the F fundamental is weakened, but the sense of B as fundamental is still not established (particularly as the pitch class B is not part of the collection). The addition of a B to the sonority in m. 18 resolves this ambiguity, and decisively shifts our sense of fundamental from F to B. The fifth B/F-sharp (one of the simplest just intervals) plays a strong role in confirming B as the root. So, now our *Tonvorstellung* of the total sonority is B(8:10:11:12:14).

As our sense of fundamental changes from F to B, the harmonic sense we make of each of the pitches also changes: for example, the E-flat stops sounding like a natural seventh above the fundamental, or F(7), and takes on the quality of a major third: B(10). The tone representation of the interval F/A changes from F(8:10), at 386 cents, to B(11:14), at 418 cents. Our tolerance for mistuning of these just intervals makes it possible to hear the equal-tempered third of 400 cents as an approximation of either interval, and as a pivot between the two fundamentals. Thus, by accepting a degree of tolerance in our tone representations, complex just intonation patterns can be conveyed in twelve-tone equal temperament.

The added D in measure 19 throws our recognition of B harmony into doubt, much as the F-sharp weakened our sense of the F fundamental in m. 16. We can persist with a B tone representation of B(16:19:20:22:24:28), but the simpler option G(10:12:13:14:15:18) is more aurally convincing—and, analogously to the arrival of B in m. 18, the G eventually appears a fifth below the D to confirm this reading. Note that the shift in fundamentals from B to G mimics the shift from A to F at the beginning of this passage.

In the bars following m. 19, Ligeti continues to add pitches more rapidly; our analysis can keep up with only a few more additions before the density of pitches overwhelms our capacity to discern a clear harmonic structure. After this point, a motivic or transformational analysis (like Bernard's analysis using flip-spin-glide operations) could better describe the music's progress. The tone representations discussed here can easily coexist with such atonal, piece-specific approaches to analysis. Motivic and atonal interval structures may also take part in purely harmonic processes, which tone representation can describe in detail.

Unlike motivic or transformational analyses, which focus on unique *intraopus* relationships such as the repetition or transformation of characteristic motives or pitch collections, this analysis has used tone representation to describe how the unfolding of pitches creates a sense of shifting just intonation sonorities. In place of unique, piece-specific relationships, the set of just intervals is taken as an *interopus* constant governing our harmonic perception. That is, instead of assuming a completely atonal world, in which the only landmarks are motivic correspondences within a work, an analysis using tone representation posits a world of extended tonality governed by a consistent set of just harmonic relationships.

Tone representation in Schoenberg's Op. 11, No. 2, "chorale"

In choosing one tone representation over another, we are not merely dealing with abstractions—these choices have something concrete to say about our musical understanding of each pitch in the collection and its relationship to all the others. When we perceive a diminished fifth as 11/8 instead of 7/5, it has different tonal implications and as Riemann notes, an "entirely different expressive value, character and content." A change in the understood root changes the meaning of each of the chord tones—a pitch that's relatively stable in one reading can become exotic and harmonically distant in another. Even if we do not precisely agree on the correct tone representation for a given pitch set, the terminology introduced here offers a way of discussing what we hear—

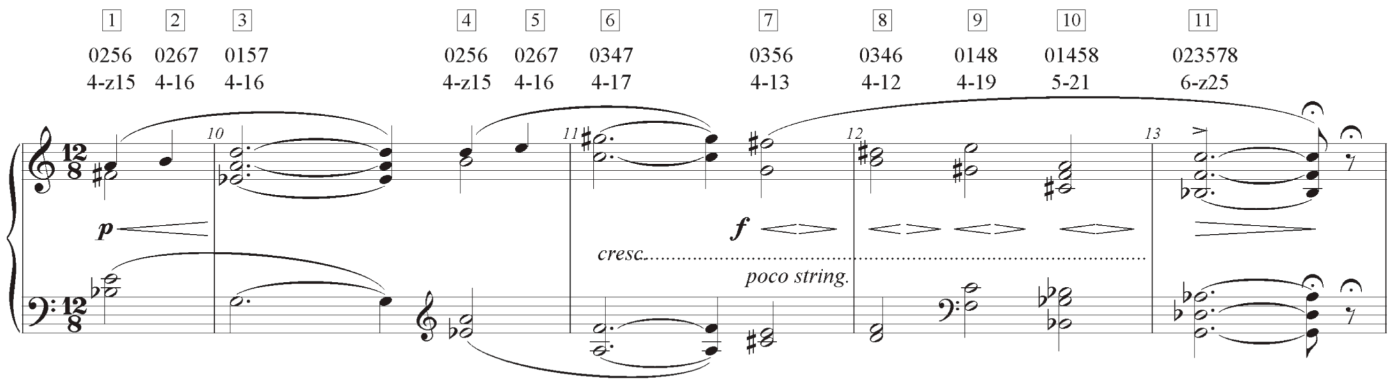
asserting one tone representation over another is a meaningful, and above all, a *musical* activity. We can continually test our analyses by playing potential roots under a harmony, or experimentally adding pitches to see how they strengthen or weaken our hypotheses.

This technique offers an alternative to existing analytical methods for works from the atonal repertoire. Unlike pitch-class set analysis, which focuses on motivic relationships *between* chords, tone representation allows us to closely examine the tensions within a single harmony in a way that's sensitive to vertical spacing and the delicate balance of different tonal implications. This sensitivity is particularly valuable for the music of the last century, in which striking individual sonorities are such an important feature. Rather than attempting to illustrate an organic coherence through the repetition of identical harmonic motives, we can discuss changing color and harmonic "rootedness." We don't need to compare these chords to one another to get at their internal tensions and qualities, since we can refer to a consistent interpretive strategy based on the overtone series instead. This approach allows us to discuss post-common-practice harmony from a phenomenological, rather than an organicist viewpoint. We can drop the assumptions of organic unity through varied repetition and still have much to say about harmony! Our intuitions about relative consonance or dissonance (as complexity of ratio) are well described by this kind of tone representation, as are our intuitions of harmonic roots.

Figure 2.16a is the "chorale" from Schoenberg's Piano Piece, op. 11, no. 2. Because the tetrachords of the chorale fall into different set classes, this passage poses a challenge to pitch class set analysis. David Lewin has discussed this passage at length, using Klumpenhouwer networks to answer the question "Is there some way in which we can sense the harmonic field of the phrase as unified, rather than diverse?" He sets out an agenda for analysis: to relate the tetrachords of different pc-set class into a unified overall view, which includes the five- and six-note sets at the end of the phrase.[51]

[51] David Lewin, "A Tutorial on Klumpenhouwer Networks, Using the Chorale in Schoenberg's Opus 11, No. 2," *Journal of Music Theory* 38/1 (1994): 79-86.

Example 3a: The "chorale" in Schoenberg's Op. 11, No. 2, with set classes labeled



Example 3b: Table of plausible tone representations for each chord—most convincing interpretations in boldface

1	B♭ E F♯ A	*distance*
A -3¢	(17:24:27:32)	5.41
G +1¢	(19:27:30:36)	15.86
E♭ -8¢	(12:17:19:23)	27.49
F♯ +12¢	**(10:14:16:19)**	**28.40**
C +38¢	(7:10:11:13)	40.21
E -21¢	(11:16:18:21)	49.94
B♭ +5¢	(8:11:13:15)	73.50

2	B♭ E F♯ B	
A -4¢	(17:24:27:36)	3.36
G +6¢	(19:27:30:40)	18.04
B +10¢	(15:21:24:32)	28.10
F♯ +19¢	**(10:14:16:21)**	**28.39**
E♭ +6¢	(12:17:19:25)	29.49
C +27¢	(7:10:11:15)	35.43
E +11¢	(11:16:18:24)	50.74
B♭ +1¢	(8:11:13:17)	73.34

3	G E♭ A D	
C -2¢	(12:19:27:36)	7.15
G -12¢	**(8:13:18:24)**	**38.70**
E♭ +19¢	**(5:8:11:15)**	**41.99**

4	E♭ A B D	*distance*
D -3¢	(17:24:27:32)	5.41
C +1¢	(19:27:30:36)	15.86
A♭ -8¢	(12:17:19:23)	27.49
B +12¢	**(10:14:16:19)**	**28.40**
F +38¢	(7:10:11:13)	40.21
A -21¢	(11:16:18:21)	49.94
E♭ +5¢	(8:11:13:15)	73.50

5	E♭ A B E	
D -4¢	(17:24:27:36)	3.36
C +6¢	(19:27:30:40)	18.04
E +10¢	(15:21:24:32)	28.10
B +19¢	**(10:14:16:21)**	**28.39**
A♭ +6¢	(12:17:19:25)	29.49
F +27¢	(7:10:11:15)	35.43
A +11¢	(11:16:18:24)	50.74
E♭ +1¢	(8:11:13:17)	73.34

6	A F C G♯	
F +4¢	**(5:8:12:19)**	**13.99**
A -7¢	(8:13:19:30)	46.38

7	C♯ E G F♯	*distance*
F♯ +6¢	**(12:14:17:32)**	**33.50**
A +9¢	**(10:12:14:27)**	**33.75**
A +26¢	(5:6:7:13)	53.52

8	D F B D♯	
D -2¢	(16:19:27:34)	7.98
E♭ +9¢	(15:18:25:32)	28.11
G +18¢	**(12:14:20:25)**	**30.05**
B♭ +9¢	(10:12:17:21)	31.53

9	F C G♯ E	
F +3¢	**(8:12:19:30)**	**11.59**
B♭ +19¢	(6:9:14:22)	50.97

10	B♭ G♭ B♭ C♯ F A	*distance*
G♭ +5¢	**(5:8:10:12:15:19)**	**16.16**
B♭ -6¢	(8:13:16:19:24:30)	40.50

11	G D♭ A♭ B♭ F C	
A +14¢	(7:10:15:17:25:38)	34.04
G +18¢	**(8:11:17:19:28:42)**	**53.21**
F -3¢	(9:13:19:21:32:48)	54.81
D♭ +22¢	(11:16:24:26:40:60)	63.14
E♭ -3¢	(5:7:11:12:18:27)	67.17

Figure 2.16
a: The "chorale" in Op. 11, No. 2, with set classes labeled
b: Table of plausible tone representations for each chord—most convincing interpretations in boldface

Unlike pitch-class set analysis, which tends to emphasize motivic relationships *between* sonorities, tone representation makes it possible to discuss the competing root implications and inner harmonic tensions within a single chord. Figure 2.16b lists several plausible tone representations of each chord of the chorale, with the most convincing representation marked by boldface type. (On a few occasions, two representations seem equally convincing, and both are in boldface.) In this table, I've listed only the best matches—based on closeness of fit and simplicity of intervals—from the list of possibilities produced by computer calculation. A typical situation can be seen in my analysis of the first chord, B-flat E F-sharp A. The tone representation which entails the least retuning is $A_{-3¢}$(17:24:27:32)—according to this representation, B-flat is heard as the seventeenth partial of a notional low A fundamental (lowered by 3 cents from equal temperament), E is heard as the twenty-fourth partial, and so on. While the intonational fit is very precise—with a distance of only 5.41 between the heard chord and its just intonation representation—the intervallic relationships between the pitches are complex and obscure. For example, we're asked to hear the interval from B-flat to F-sharp as the exotic interval 17:27, though we would intuitively prefer a simpler interpretation like 5:8, a just minor sixth. We can find a simpler interpretation of the whole tetrachord by accepting a greater distance between the heard set and its just intonation representation. The tone representation F-sharp$_{+12¢}$(10:14:16:19) provides the most convincing compromise between intonational accuracy and simplicity of interval ratio—the inclusion of the fourth octave of the fundamental (16) further strengthens its appeal.

A complete discussion of Figure 2.16 is impractical here, but I would like to point out a few general observations which illustrate how the theory of tone representation might contribute to an analytical reading. When we look at the most likely roots for each chord in the passage, we see the frequent repetition of just a few pitch classes (allowing for some variability of tuning): F, F-sharp/G-flat, and G. These three pitches account for nine of the eleven chords of the chorale. G is the root of Chord 3 at the end of the first gesture, as well as Chord 8 and the cadential Chord 11, although the chords differ in cardinality and set class. An upward progression by semitone from one "fundamental bass" pitch class to the next recurs frequently—first as F-sharp to G from Chords 2 to 3, then as F to F-sharp to G in Chords 6, 7, and 8. This "fundamental bass" progression is repeated in Chords 9 to 11 as F to G-flat to G, even though the pitch content of the chords is different. The fundamental bass progression cuts across the phrase structure in an interesting way, inviting the listener to group Chords 6, 7, and 8 across the notated phrase boundary between Chords 6 and 7. This is one way we could make sense of the crescendo beginning below Chord 6—the increase in intensity accompanies the beginning of the ascent in the fundamental bass.

In a different hearing of the passage, we can hear the boundaries between the phrases as revoicings of the harmony over a repeated fundamental bass—thus, both Chord 3 and Chord 4 can be heard as rooted on E-flat, while Chords 6 and 7 share a root of A. Note that this reading interprets the roots of these chords differently than the previous analysis—the divergent interpretations reflect two possible ways of hearing the structure of the passage, which is rich and complex enough to support a range of competing analyses.[52]

[52] The analysis offered here might seem to imply that Schoenberg's music is "out of tune" and needs to be "corrected" to just intonation: this is not my intent, and Schoenberg makes his preference for equal temperament abundantly clear. Rather, I'd argue that equal temperament allows the careful balancing of ambiguities between several tone representations, an ambiguity that seems integral to the aesthetic of early

By invoking tone representation, we are no longer treating this music as atonal, but rather as exhibiting a kind of extended tonality. As we've seen, the ratio model focuses our attention on very different aspects of pitch structure than those illuminated by distance-based models like pitch-class set analysis. I do not deny the utility of atonal theories like pitch-class set analysis for this repertoire, but they are designed to describe different kinds of relationships than those I'm interested in exploring here. In a sense, no music is truly atonal—there is music in which atonal relationships are the basis of convincing analytical interpretations, but this does not rule out the possibility of other tonal or quasi-tonal readings. If we do not insist on shoehorning musical works into the structures of just one theory at a time, the two methods could be usefully combined—atonal theory's emphasis on motivic transformation could be complemented by tone representation's attention to vertical spacing, sonic color, and implied roots.

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atonality. A similar conclusion is drawn by Gary Don in his research on overtone series chords in the music of Debussy: he concludes that Debussy "was content to incorporate the overtone series into his music through the lens of equal temperament, thus suggesting a particular sonority, without requiring a literal (i.e., just intonation) realization of those sonorities." See Don, "Brilliant Colors Provocatively Mixed: Overtone Structures in the Music of Debussy," *Music Theory Spectrum* 23/1 (2001): 61-73.

CHAPTER 3: Extended Just Intonation in Theory and Practice

Introduction: Precursors of extended just intonation

Of all the neglected chapters of music theory, just intonation seems at first glance the least likely to resurface in the century of serialism and chance composition. The essential principle of just intonation is that simple ratios between numbers produce the ideal, pure tunings for intervals in tonal music; with the standardization of equal temperament and the advent of atonality, just intonation seemed fated to obscurity and irrelevance. What use could composers have for an ancient tuning system, designed for simple tonal structures, that lacked the ease and flexibility of equal temperament?

For Harry Partch (1901-1974), the most influential figure in the just intonation revival in America, the appeal of this antiquated practice was thoroughly modern: the desire to build a rational musical world from strict rational principles, rather than simply accepting an established practice now alienated from its origins. For Partch, the music theory of a thriving culture would be built on a solid bedrock of "Archean granite" instead of on the inherited instruments, forms, and tunings of eighteenth-century Europe.[1] Just intonation reflected the natural preferences of the listening ear: on this solid foundation, he saw the possibility for an controlled, logical expansion of musical resources, eventually breaking free of the "prison bars" of the twelve-tone keyboard.[2]

By turning to just intonation as the basis for his theories, Partch placed himself in a tradition with a long history, which we can divide into three basic stages: Pythagorean,

[1] Harry Partch, *Genesis of a Music: An Account of a Creative Work, Its Roots, and Its Fulfillments,* second edition (New York: Da Capo Press, 1974): xvii.

[2] Harry Partch, "Bitter Music," in *Bitter Music: Collected Journals, Essays, Introductions, and Librettos* (Urbana, Illinois: University of Illinois Press, 1991): 12.

just, and extended just intonation. *Pythagorean intonation* refers to the tuning theory of the ancient Greeks, invented (as tradition has it) by Pythagoras in the sixth century B. C. and devoutly defended until the fifteenth century (and even beyond). The Greeks recognized that whole number ratios between string lengths on their monochords, based only on the multiples of 2 and 3, could define all of the intervals of their music. Simple ratios provided the foundational 2/1 (octave), 3/2 (fifth), and 4/3 (fourth). The simplicity of an interval's ratio is directly related to its consonance: the less concordant intervals of Greek theory, like the 9/8 whole tone or the 256/243 *limma,* were formed by more complex ratios including higher multiples of 2 and 3.

Pythagorean intonation dominated European music theory until the Renaissance, when theory came into serious conflict with the practice of composers and performers. The problem was this: musicians had begun to treat thirds and sixths as consonances, but in Pythagorean tuning, these intervals are only available as complex, harsh-sounding ratios. The Pythagorean version of the major third, for example, is the thorny ratio 81/64, which is harmonically unstable and difficult to sing. The solution proposed by Bartolomeus Ramis de Pareia in 1482 was simple but had far-reaching effects: by admitting the number 5 and its multiples into the tuning system, the jarring ratio 81/64 could be replaced by the smooth, mellow 5/4. The use of 5 also provided simple ratios for the minor third (6/5), minor sixth (8/5), and major sixth (5/3).

This tuning system, based on ratios containing multiples of 2, 3, and 5, is known as *just intonation.* Despite its admirable acoustic purity, just intonation leads to some practical difficulties, especially when implemented on keyboard instruments, which are incapable of making the fine continuous adjustments of a violin or trombone. It is impossible to tune the twelve-note-per-octave keyboard of European tradition in just intonation in one key without creating drastically out-of-tune notes in other keys. To cope with this problem, various *temperaments* have been applied to the keyboard. The current standard is equal temperament—every step on the keyboard is made the same size, exactly one-twelfth of an octave. Instead of simple integer ratios, each interval except the octave is based on an irrational number, the twelfth root of 2. Compared to the ideal just intervals, each interval is slightly out-of-tune, but still usable. In the nineteenth and twentieth centuries, the harmonic flexibility of equal temperament outweighed the pure sound of just intonation, and equal temperament became the standard tuning for all keyboard instruments.

While Partch found the natural acoustical and physiological roots of just intonation a suitably "natural" foundation for his musical system, the intervallic palette of just intonation (established in the fifteenth and sixteenth centuries) was not sufficient for the complexities of his flowing, speech-like music. Partch found it necessary to go beyond the intervals based on 2, 3, and 5 to higher prime numbers including 7 and 11: this modification introduced *extended just intonation.* To describe the prime numbers included in a tuning system, Partch introduced the useful idea of "limits"—the limit of an intonational system is the highest prime number which is a factor in its interval ratios. Thus Pythagorean intonation has a "three-limit" and Renaissance just intonation a "five-limit": Partch's own tuning system reached the eleven-limit. For Partch, the extension of just intonation to the higher prime numbers was part of a historical imperative: from earliest times, he writes, the use of musical materials has "progressed from the unison in the direction of the great infinitude of dissonance."[3] The sense of historical destiny in Partch's thought is shared by other extended just intonation advocates: James Tenney argued that due to the limitations of equal temperament, the development of harmony in Western music came to a halt in the early twentieth-century, and can only be restarted through the exploration of the intervallic possibilities of the higher primes.[4]

Earlier theories including these higher primes predate Partch's work, though no previous thinkers explored this territory so systematically or ambitiously. The second-century Greek theorist Ptolemy, one of Partch's historical exemplars, proposed monochord divisions including intervals with prime factors as high as 11, though how accurately this reflected actual musical practice is unknown.[5] The prime number 7 is proposed as the basis of the dominant seventh by eighteenth-century theorists Georg Andreas Sorge and Leonhard Euler, despite the intonational problems of fitting this seven-limit interval into the standard five-limit diatonic scale. The possibility of extending just intonation to multiples of seven is also considered in Helmholtz's *On the Sensations of Tone;* this work is particularly important to the development of extended just intonation in the twentieth century, as his theories of acoustics and harmonic perception provide the underpinning for Partch's theories.[6] (American composer Ben

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[3] Partch, *Genesis of a Music,* 94.

[4] James Tenney, "Reflections after Bridge," liner notes to *James Tenney: Bridge & Flocking.* Hat Hut, ART CD 6193, 1996.

[5] See Partch, *Genesis of a Music,* 368-69.

[6] Hermann Helmholtz, *On the Sensations of Tone,* second edition (London, Longmans and Co., 1885). Translation by Alexander Ellis of *Die Lehre von den Tonempfindungen,* fourth edition. Helmholtz notes that the seventh of the dominant seventh chord "so nearly corresponds to the corresponding partial tone in the compound tone of the dominant, that the whole chord may be very well regarded as a representative of that compound." He suggests that due to this, the dominant seventh is the "softest of all dissonant chords." (347) The footnotes provided by Alexander Ellis, the English translator of *On the Sensations of Tone,*

Johnston also credits an early exposure to Helmholtz's research as a spur toward his own work in extended just intonation.)

We can identify two main tendencies in the twentieth-century just intonation revival: one is the rejection of equal temperament and its tuning compromises for a purer, more authentic tuning system; the second is the exploitation of new, more complex harmonies based on the higher prime numbers of extended just intonation—that is, the intervals to be found between the upper overtones of the harmonic series. These two tendencies are not always present at the same time: for example, many composers of the early twentieth century showed eager interest in the new harmonies implied by the higher overtones, but were still unwilling to give up the advantages of equal temperament. Similarly, we find just-intonation composers happily working within the Renaissance five-limit for its clarity of intonation and purity of sound, and eschewing the complexities of the higher primes.

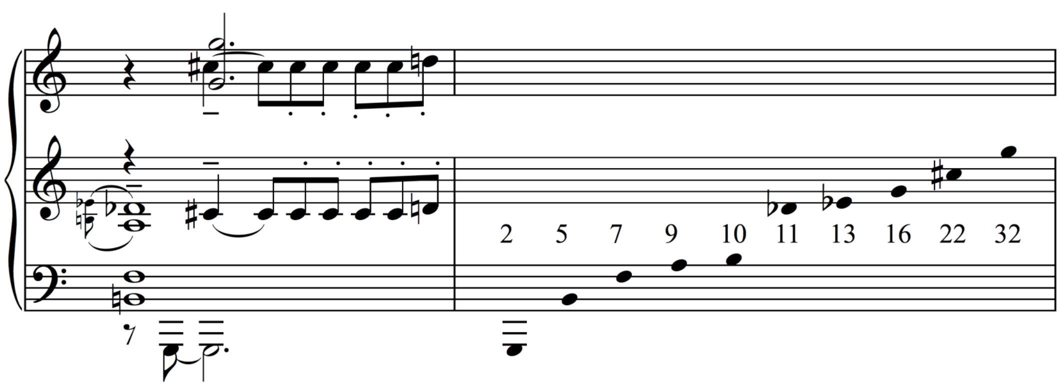

Figure 3.1: Excerpt from Claude Debussy's *Canope,* Preludes, Book II (1912-1913)

The music of Claude Debussy provides many instances of equal-tempered harmonies which evoke the upper overtones: Figure 3.1 analyzes an excerpt from the prelude *Canope* (1912-13). Here, the whole-tone scale G A B D-flat E-flat F is arranged

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would presumably have interested Partch greatly: Ellis frequently refers to aural tests on the "Harmonical," a retuned harmonium in just intonation including the natural seventh (17n).

to imply an overtone series on a G an octave below the actual bass (below the range of a standard piano): the sounding bass is the second partial of this low G, B the fifth, F the seventh, A the ninth, D-flat the eleventh, and E-flat the thirteenth—all approximated as closely as possible in equal temperament. Such overtone structures can also be found in music by Ravel, Scriabin, and even Webern and Berg, as demonstrated in recent research by Olli Väisälä and Gary Don.[7]

The relation of the overtone series to new harmonic developments in the early twentieth century became a staple of theoretical writing, as seen in writings by Arnold Schoenberg and Paul Hindemith. Schoenberg takes on Helmholtz's characterization of the relation between consonances and dissonances as a continuum, not a sharp division into categories: "They are no more opposites than two and ten are opposites, as the frequency numbers indeed show; and the expressions "consonance" and "dissonance," which signify an antithesis, are false."[8] Like Partch and Tenney, Schoenberg sees music evolving to gradually embrace the higher overtones: music "has drawn into the stock of artistic resources more and more of the harmonic possibilities inherent in the tone."[9] In his 1934 essay, "Problems of Harmony," Schoenberg outlines how the chromatic scale might be understood as the combination of the upper overtones of the tonic, subdominant

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[7] Olli Väisälä, "Prolongation of Harmonies Related to the Overtone Series in Early-Post-Tonal Music," *Journal of Music Theory* 46/1-2 (2002): 207-283 and Gary Don, "Brilliant Colors Provocatively Mixed: Overtone Structures in the Music of Debussy," *Music Theory Spectrum* 23/1 (2001): 61-73. See also Célestin Deliège, "L'harmonie atonale: de l'ensemble à l'echelle," in *Sources et ressources d'analyses musicales: journal d'une démarche* (Sprimont: Mardaga, 2005): 387-411.

[8] Arnold Schoenberg, *Theory of Harmony* (Berkeley, California: University of California Press, 1978): 21. Translation by Roy Carter of *Harmonielehre,* third edition (Vienna: Universal Edition, 1922).

[9] Schoenberg, op. cit., 21.

and dominant: Figure 3.2 illustrates the overtones from 1 to 13 of the pitches C, F, and G.[10]

1	2	3	4	5	6	7	8	9	10	11	12	13
C	C	G	C	E	G	B-flat	C	D	E	F-sharp	G	A-flat
F	F	C	F	A	C	E-flat	F	G	A	B	C	D-flat
G	G	D	G	B	D	F	G	A	B	C-sharp	D	E-flat

Figure 3.2: Schoenberg's derivation of the chromatic scale from the overtone series

The combination of overtones 1 to 6 of each fundamental build the pitches of the white-note diatonic scale: the five chromatic degrees are derived from overtones 7, 11, and 13 (sometimes with in multiple versions—C-sharp/D-flat can be the eleventh harmonic of G or the thirteenth of F, and E-flat can be either the seventh of F or the thirteenth of G). For these harmonic relationships to hold, one must accept a substantial degree of mistuning: the 11th harmonic is 49 cents below its tempered approximation as a tritone, and the 13th harmonic is 41 cents above the equal-temperament minor sixth.[11] The overtone series, approximated to equal temperament, is also featured in Hindemith's *The Craft of Musical Composition.*[12] Hindemith ranked chords and intervals on a scale of varying tension, based on a study of each interval's combination tones. Chords could then be assigned roots, based on the most consonant interval. Like Schoenberg, Hindemith suggests that chords in equal temperament can best be understood by an appeal to an underlying acoustic framework; this framework has much in common with the complex ratios of extended just intonation.

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[10] Schoenberg, "Problems of Harmony," in *Style and Idea: Selected Writings of Arnold Schoenberg* (New York: St. Martin's Press, 1975): 271.

[11] Joseph Yasser challenges Schoenberg on these tuning problems in an exchange published as "A Letter from Arnold Schoenberg," *Journal of the American Musicological Society* 6/1 (Spring 1953): 53-62. Yasser advocated a 19-tone division of the octave in his *Theory of Evolving Tonality* (New York: Da Capo Press, 1975).

[12] Paul Hindemith, *The Craft of Musical Composition* (New York, Associated Music Publishers, Inc., 1941). Translation of *Unterweisung in Tonsatz* (Mainz: B. Schott's Söhne, 1937).

Henry Cowell's 1930 *New Musical Resources* asserts the importance of upper overtones in the perception of harmony: like Schoenberg, he argues that the history of harmonic development has been a rise gradually upward to embrace the higher overtones.[13] As in Schoenberg's thought Cowell's scale from consonance to dissonance is relative: the acceptance of a sonority as one or the other depends on the musical education and acculturation of the listener. Cowell suggests that chords in equal temperament can best be understood as approximations of extended just intonation combinations (21), in the spirit of the Debussy analysis in Figure 3.1 above. More complex harmonies may be the result of polychords, the mixture of pitches corresponding to overtones of two or more fundamentals.

Cowell also invokes the idea of undertones, which was central to the dualist theories of German nineteenth century theorists Arthur von Oettingen and Hugo Riemann. In Riemann's theories, the undertone series is imagined as an exact inversion of the overtone series: thus, while the major triad is composed of the fourth, fifth, and sixth overtones, the minor triad is composed of the fourth, fifth, and sixth *undertones,* with the traditional fifth of the minor triad as the generator. While undertones are not actually created by vibrating bodies in the same way as overtones, we could imagine the undertone series as all the pitches which *have* the generator as *their* overtone. This distinction was explained by Oettingen: the pitches of the minor triad have *(haben)* a common overtone; the pitches of the major triad are *(sein)* overtones of a common fundamental.[14] Theories invoking the undertone series are most successful when they

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[13] Henry Cowell, *New Musical Resources* (New York: Knopf, 1930): 3.

[14] Arthur von Oettingen, *Harmoniesystem in dualer Entwickelung* (Dorpat and Leipzig: Gläser, 1866).

keep this distinction clear: the essential difference between overtones and undertones confounds attempts to use them in exact symmetry with one another.

The explication of just intonation in *New Musical Resources* is marred by a general vagueness in pitch measurements: pitches tend to be rounded off to the nearest equal-temperament note. This liberal rounding off means that for Cowell there is little difference between the natural seventh-based 8/7 "major second" and the 10/9 and 9/8 major seconds. Cowell never converts ratios into cents, which might have suggested a greater precision and a clearer recognition of the differences between just and tempered intervals. The rhythmic innovations in *New Musical Resources* are far more thoroughly worked out than the analogous pitch systems (and play a greater role in Cowell's own compositions): Cowell explores divisions of the basic bar or beat into each of the odd numbers 1 to 15, suggesting a metaphorical relationship to the overtone series: this prefigures experiments with tempo by composers ranging from Stockhausen to Nancarrow. Cowell also introduces unusual meters like 6/9 and 2/6, and suggests that ratios between rhythmic values can be understood as analogous to vibration ratios in just intervals.[15]

Harry Partch

While Debussy, Schoenberg, Hindemith, and Cowell all speak of drawing expanded harmonic resources from the unexplored portions of the overtone series, these composers never cut their ties to the established twelve-tone system. This step fell to Harry Partch, whose music and theoretical treatise *Genesis of a Music* (1949/1974) are

[15] See Kyle Gann, "Subversive Prophet: Henry Cowell as Theorist and Critic," in *the Whole World of Music: A Henry Cowell Symposium,* ed. David Nicholls (Amsterdam: Harwood Academic Publishers, 1997): 176.

the departure points for extended just intonation in the twentieth century. Partch's book is unique in its development of a new musical language—including harmony, notation, and even instruments—from just a few basic principles.

Beginning in the 1920s, Partch created new instruments to play the extended just intervals unavailable in the traditional orchestra. His first constructions were an adapted viola, with an elongated fingerboard marked with positions for extended just intervals, and a retuned reed organ, which he named the "Ptolemy" (later to be rechristened the "Chromelodeon" in a rebuilt version). Part of the pleasure of reading *Genesis of a Music* (apart from Partch's opinionated and vigorous writing style) are the illustrations and names of his instruments, including the Cloud Chamber Bowls, the Spoils of War, the Bloboy, and the Eucal Blossom.

Partch's *Genesis of Music* is organized around two concepts. One is Corporealism—"the essentially vocal and verbal music of the individual" (8); Partch opposes Corporeal music, linked to poetry and dance, to the abstraction of Western instrumental art music. Partch's second guiding concept is Monophony: "an organization of musical materials based upon the faculty of the human ear to perceive all intervals and to deduce all principles of musical relationship as an expansion from unity" (71). In the Monophonic organization of pitch, Partch is always conscious of a single pitch—or to be more accurate, pitch class—representing "unity"; in more traditional musical terms, we would term this a tonic, though Partch prefers to designate it by the ratio 1/1. Thus far, we've discussed ratios as representing *intervals,* not *pitch classes.* The ratio 9/8, for example, specifies the interval of the major whole tone (about 204 cents) between two pitches, but not precisely what those pitches are. In Partch's Monophonic pitch world, ratios represent a pitch's relationship to the central pitch 1/1. Thus, 9/8 means the pitch a whole tone above 1/1, 4/3 represents the pitch a fourth above, and 3/2 the pitch a perfect fifth above. In theory, the inverse of these ratios would refer to pitches the same distance *below* 1/1: thus 8/9 would be a whole tone below 1/1, 3/4 a fourth below, and 2/3 a fifth below. For a simplified set of pitch class names, though, Partch prefers to transpose these pitches by octave into the octave from 1/1 to 2/1: instead of 8/9, a tone below 1/1, he typically uses 16/9, a minor seventh above 1/1; instead of 3/4 (a fourth below), he uses 3/2, and instead of 2/3, 4/3. Monophony and Corporealism were once found together, Partch suggests, in ancient Greece where "ratio-idea and music-enhanced word vitality" were present in the same musical culture (60). The restoration and reunion of these concepts in the modern world are the goal of Partch's theorizing and composition.

In *Genesis of a Music,* Partch describes a 43-tone scale chosen from intervals within the 11-limit within a single octave (from 1/1 to 2/1).[16] Figure 3.3 shows the diagram of this scale that Partch nicknamed the "One-Footed Bride". The diagram is to be read beginning with the ratio 1/1 at the lower left; pitch ascends up the left column of ratios from 1/1 to the tritone exactly half an octave above (not expressible as an integer ratio). The note names from G to C# mark this ascent. We must read the right side of the diagram from the top down—this reversal puts each interval directly opposite its inverse, with which it shares a similarity in sound and relative consonance. The fourth 4/3 is opposite the fifth 3/2, the major third 5/4 opposite the minor sixth 8/5, and so on.

[16] Partch, *Genesis of a Music,* 133. As Partch scholar Bob Gilmore has documented, Partch experimented with scales of different sizes throughout his career, all selected from the boundless Monophonic "fabric" of intervals: see "Changing the Metaphor: Ratio Models of Musical Pitch in the Work of Harry Partch, Ben Johnston, and James Tenney." *Perspectives of New Music* 33/1-2 (Winter-Summer 1995), 458-503.

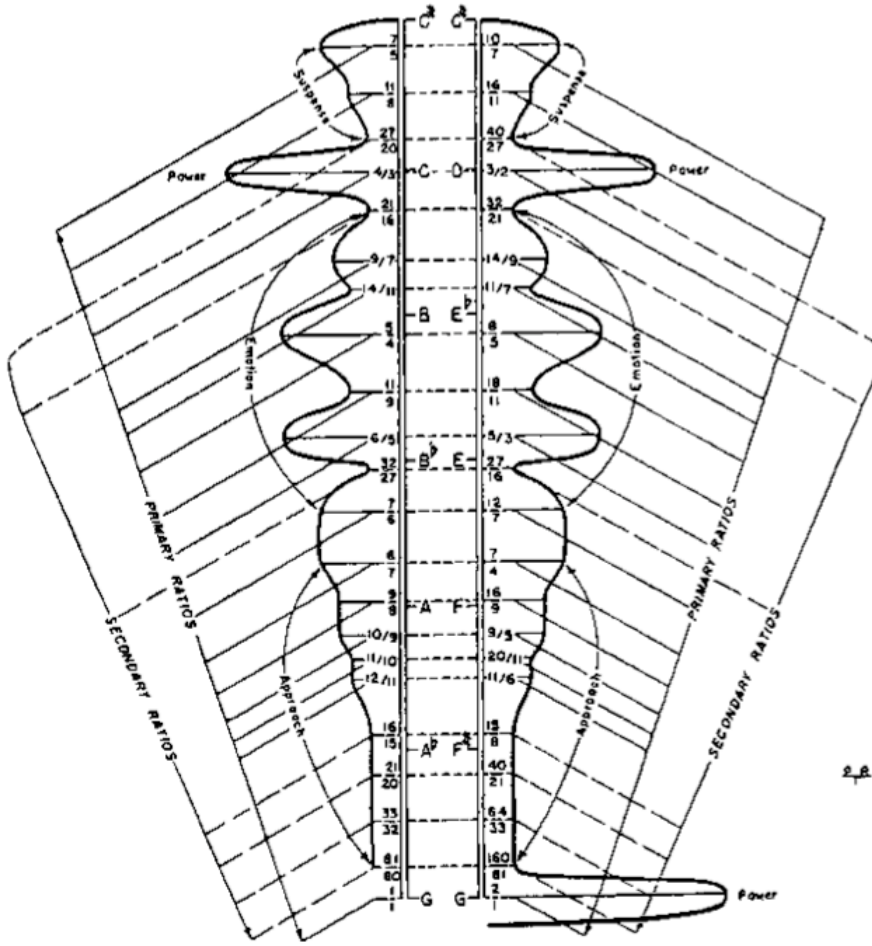

Figure 3.3: "The One-Footed Bride: A Graph of Comparative Consonance"[17]

Note that while all the pitches in the scale are related by just intervals to 1/1, not all are expressible as *overtones* of 1/1. For example, the minor third of the scale, 6/5, would only fit into the overtone series of a fundamental a just major third below 1/1; similarly, the fourth 4/3 would require a fundamental a fifth below 1/1. When pitches can be found in the overtone series of 1/1, they are called "Identities"—thus 3/2 is an example of the 3-Identity, and 5/4 is the 5-Identity. Most of the intervals are "primary ratios" built from the integers 1 to 11; dotted lines point out "secondary ratios" built from multiples of these primary integers including 15, 21, 27, 33, and 81.

[17] Partch, *Genesis of a Music,* 155.

Partch draws a curve alongside the columns of intervals to show each one's relative consonance; this curve gives the "bride" her distinctive profile. A point on the curve far from the central column represents a high degree of consonance (peaking at the octave, the bride's "foot"); a point on the curve close to the central column indicates a more dissonant interval; for example, the intervals smaller than a semitone at the bottom of the column or the complex 27/20 near the top.

The One-Footed Bride also categorizes the pitches into four types based on the interval they form with 1/1: intervals of approach (the 231 cent 8/7 septimal whole tone or smaller), intervals of emotion from the small 7/6 minor third (267 cents) to the flat fourth 21/16 (471 cents), intervals of power (the perfect fourth and fifth), and intervals of suspense (diminished fifths/augmented fourths).

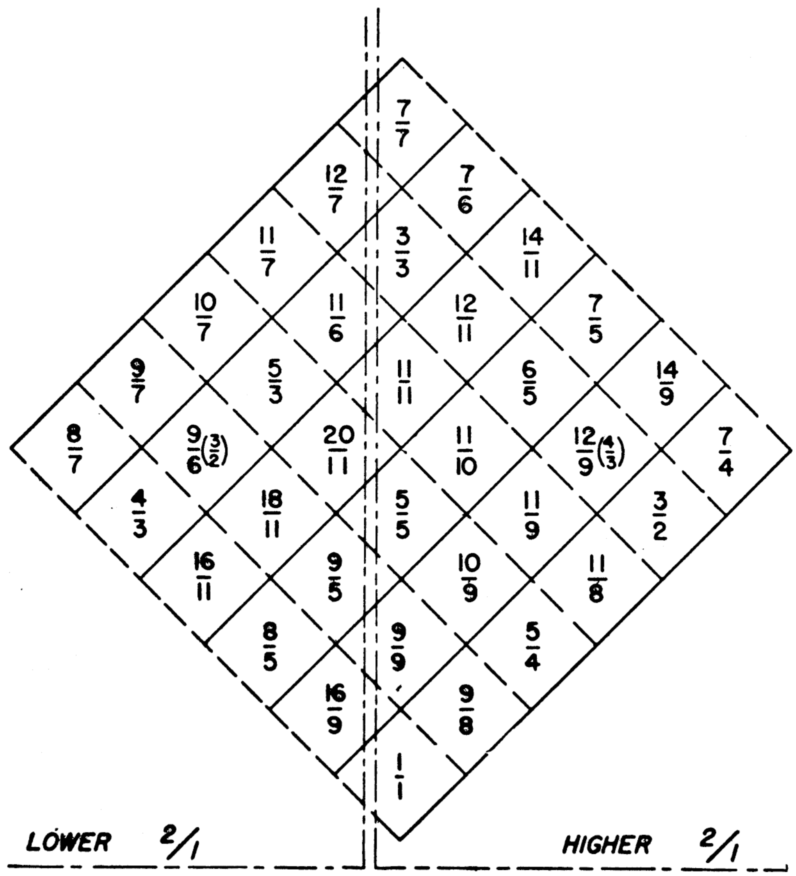

Figure 3.4: "The Expanded Tonality Diamond"[18]

The melodic scale diagrammed in the "One-Footed Bride" takes on a harmonic dimension in Partch's Tonality Diamond (Figure 3.4). The tonality diamond is a matrix of overlapping "Otonalities" (six note sets based on the odd overtones 1 through 11) and their inversions, "Utonalities" (based on the odd *undertones* 1 through 11). Figure 3.5 shows how the members of each Otonality/Utonality fall in ascending/descending order within an octave (cent values show the pitch class relative to 1/1 = 0):

[18] Partch, *Genesis of a Music,* 159.

Otonality:	1/1	9/8	5/4	11/8	3/2	7/4
cents:	*0*	*204*	*386*	*551*	*702*	*969*
Utonality:	1/1	16/9	8/5	16/11	4/3	8/7
cents:	*0*	*996*	*814*	*649*	*498*	*231*

Figure 3.5: Pitch content of Otonalities and Utonalities

The Tonality Diamond lays out the twelve harmonic areas—six Otonalities and six Utonalities—which include 1/1 in as a member. 1/1 is alternately given 12 meanings, as different Odentities of Otonalities and Utonalities. In the Tonality Diamond, Otonalities are read from lower left to top right (within the solid lines), and Utonalities are read from lower right to top left (within the dotted lines). Thus, the Otonality on 8/7 starts at the leftmost corner, and continues to 7/7 (equal to 1/1) at the top of the diamond; the Utonality on 7/4 reads from the rightmost corner to the top of the diamond in the opposite direction. Up the center of the diagram runs a series of ratios, all equivalent to 1/1: pitches to the left of this spine are in a lower octave (or 2/1, in Partch's preferred terminology) and pitches to the left are in the higher octave above 1/1. The resemblance of this diamond to the matrices of prime and inverted forms in serial music is notable, though not specifically acknowledged by Partch. Every pitch in the matrix is at the intersection between a Utonality and Otonality, allowing facile shifts between the two tonal types. Partch sees the unifying relation of all twelve harmonies to 1/1 as one of the great strengths of Monophony, even though this unity comes at the cost of free modulation as in equal temperament.

To complete the 43-tone scale of Figure 3.3, Partch adds 14 new pitches to the 29 distinct pitches of the tonality diamond, so that steps between adjacent scale degrees vary from between 14.4 and 38.9 cents. The inclusion of these added intervals makes possible a number of "secondary tonalities," which fall outside the Diamond because they do not include 1/1. Many of these secondary tonalities are deemed incomplete because they call for intervals falling outside the 43-tone scale; but nonetheless, they offer some expanded tonal resources beyond the twelve tonalities of the diamond.

While Partch is extremely specific about the layout of his tuning system, he has less to say about the specific application of these concepts in composition. Numerous historical scales, including the just diatonic and Pythagorean scales, are possible in various tonalities. Partch discusses how the members (Identities) of a particular Otonality or Utonality act as the six "primary planets" of that tonality; these draw nearby pitches toward themselves with a kind of gravitational pull, with the 1-identity, like a tonal root, exerting the greatest attractional force. Another harmonic device described by Partch is "tonality flux"—the interplay between nearby pitches in different tonalities. Partch offers an example juxtaposing the 4, 5, and 6 identities of the 8/7 Otonality (231, 617, and 933 cents) and the 6, 5, and 4 identities of the 7/4 Utonality (267, 583, and 969 cents)—the pitches of the second set are all within 40 cents of a pitch in the first.

Lou Harrison

Lou Harrison (1917-2003) was one of the composers most strongly influenced by Partch's *Genesis of a Music.* Harrison had studied with Henry Cowell in San Francisco and collaborated with John Cage on a number of projects in the 1930s, including several concerts of percussion music—he also studied briefly with Schoenberg in Los Angeles. Harrison was given a copy of *Genesis of a Music* in 1949 by the composer and music critic Virgil Thomson, and soon began to incorporate just intervals into his own compositions.

Unlike Partch, who wished to wipe away musical systems of the past to make a fresh start, Harrison has always been open to musical hybridization: he writes, "Don't underrate hybrid musics *because that's all there is.*"[19] Harrison's music borrows ideas from Western diatonic music as well as Asian musics from Korea to Indonesia; all of these borrowings, however, are inflected by his dictum that "Just intonation is the best intonation."[20]

In addition to many modes and pentatonic scales within the five-limit, Harrison has explored tunings which extend into the higher primes of extended just intonation. One such example is the piano retuning for *Incidental Music for Corneille's Cinna,* a suite composed between 1955 and 1957 for retuned "tack piano"—a piano with tacks inserted in the hammers for a harpsichord-like jangle. Harrison's tuning system is shown in Figure 3.6, with both ratios and values in cents. The tuning shows clear affinities to Partch's 43-tone scale, which includes all of Harrison's pitches except the 25/18 "augmented fourth." The tuning includes a variety of pure fifths and thirds, as well as septimal relationships combining B-flat (7/6) and F (7/4) with the other pitches of the scale. The harmonic writing in *Cinna* takes advantage of the ringing pure fifths of the tuning, combining them with the less familiar septimal intervals in a way reminiscent of some passages in La Monte Young's *Well-Tuned Piano.*

G	A-flat	A	B-flat	B	C	C-sharp	D	E-flat	E	F	F-sharp	G
1/1	16/15	10/9	7/6	5/4	4/3	25/18	3/2	8/5	5/3	7/4	15/8	2/1
0	*112*	*182*	*267*	*386*	*498*	*569*	*702*	*814*	*884*	*969*	*1088*	*1200*

Figure 3.6: Piano retuning in *Cinna*[21]

[19] Lou Harrison, *Lou Harrison's Music Primer* (New York: C.F. Peters, 1971): 45.

[20] Ibid., 4.

[21] See Leta Miller, ed., *Lou Harrison: Selected Keyboard and Chamber Music 1937-1994* (Madison, Wisconsin: A-R Editions, 1998): xl-xlv.

Harrison also developed a complex variant of extended just intonation he called "free style," as opposed to the "strict style" of composing with reference to a fixed scale and tonal center. In free style, melodic intervals between notes are chosen freely from a palette of just intervals: the resultant interval chains can take the intonation into distant territories very quickly.[22] Figure 3.7 is a brief excerpt from the *Simfony in Free Style.* Note that the B that begins the excerpt (assigned the cent value 1100) is not the same pitch as the B beginning the subsequent phrase in the lower system, which is 49 cents (nearly a quartertone) higher. Because of the enormous difficulty of making successive just intervals implying constantly shifting tonal centers, the work has never been received a live performance.[23]

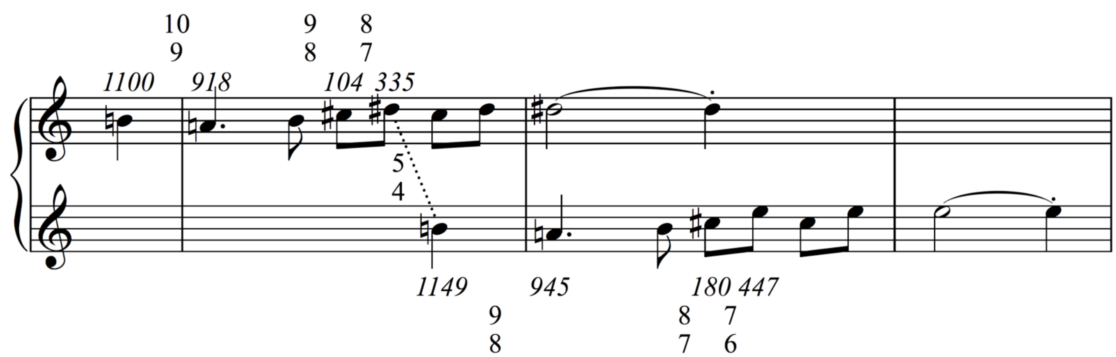

Figure 3.7: *Simfony in Free Style,* excerpt[24]

The music of the Indonesian gamelan has long fascinated Harrison, who has been a leading figure in the development of the "American Gamelan" movement. Harrison has constructed a number of gamelan, all based on his own theories of extended just intonation. The combination of gamelan with just intonation theory creates a strange

[22] Larry Polansky, "Item: Lou Harrison as a Speculative Theorist," in Peter Garland, ed., *A Lou Harrison Reader* (Santa Fe, New Mexico: Soundings Press, 1987).

[23] Other free style works include *At the Tomb of Charles Ives* and *Phrase for Arion's Leap.* See Leta Miller and Fredric Lieberman, *Lou Harrison: Composing a World* (New York: Oxford University Press, 1998): 116-121.

[24] Harrison, *Lou Harrison's Music Primer,* 6-7.

cross-cultural hybrid: just intonation has never been part of the theory behind Indonesian tunings, which are based instead on a concept of *embat,* a highly individual sense of intonation linked to the singing voice.[25] Because gamelan tunings are not traditionally fixed, many of their pitches fall close to just intervals which can be the basis for a tuning system: as Harrison's biographers Miller and Lieberman note, this imposition of just intonation on the gamelan "while not culturally *characteristic,* was culturally *possible.*"[26] Figure 3.8 illustrates a *sléndro* tuning from the gamelan Harrison built at Mills College.

octave

perfect fourth   perfect fourth

intervals between pitches: 8/7 7/6 9/8 8/7 7/6

cents: *0 231 498 702 933 1200*

Figure 3.8: Harrison's tuning for the Mills gamelan[27]

Drawing on gamelan ideas in his music for Western instruments, Harrison often borrows terms for different scale types: *sléndro* for anhemitonic pentatonic scales and *pélog* for hemitonic pentatonics. In Harrison's characterization, *sléndro* scales have wide seconds and narrow thirds, while *pélog* scales have wide thirds and narrow seconds.[28] Essential to all of Harrison's music is the idea of scale or mode: the wide currency of related concepts has made his cross-cultural borrowings compatible with one another, often to a surprising degree. With the exception of the diabolically complex "free style"

[25] The conflict between *embat* and Western "intonational naturalism" is explored by Marc Perlman in "American Gamelan in the Garden of Eden: Intonation in a Cross-Cultural Encounter," *The Musical Quarterly* 78/3 (Autumn 1994): 510-555.

[26] Leta Miller and Fredric Lieberman, "Lou Harrison and the American Gamelan," *American Music* 17/2 (Summer 1999): 146-178.

[27] Miller and Lieberman, *Composing a World,* 113.

[28] Ibid., 110. According to Perlman, *pélog* contains "seven tones that are treated less as a single scale than as a source of pentatonic scales" (op. cit., 535).

works, Harrison's use of just intonation is far less dense that Partch's: as the musicologist Bob Gilmore has pointed out, Harrison seems to use just intonation more for the uniquely limpid and transparent sound of its intervals than to add more pitches to his palette.[29]

Ben Johnston

Ben Johnston's association with Partch came not only through *Genesis of a Music,* but in the course of a six-month apprenticeship with Partch in 1950 and 1951. Johnston (b. 1926) and his wife lived in Partch's primitive studio in Gualala, California, tuning Partch's instruments and learning to play them.[30] It was, however, not until the end of the decade that Johnston began to write his own compositions in just intonation. While Partch sought to build his system from scratch, Johnston attempted to reconcile extended just intonation with the Western tradition—he composed for standard Western instruments, and was even comfortable using serial procedures in combination with just intervals. In his compositions, Johnston calls for precise tuning of complex just intervals, using a detailed system of new accidentals. Among the extended just intonation composers, Johnston is one of the most articulate theorists: his thoughts on just intonation appeared in a series of influential academic articles.[31]

At the root of Johnston's theorizing is the argument that just intervals are more *intelligible* than tempered ones: thus just intonation is a relatively more *efficient* way of

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[29] Bob Gilmore, "The Climate Since Harry Partch," *Contemporary Music Review* 22/1-2 (March/June 2003): 21.

[30] Heidi von Gunden, *The Music of Ben Johnston* (Metuchen, New Jersey: Scarecrow Press, 1986): 11-13.

[31] See especially "Scalar Order as a Compositional Resource," *Perspectives of New Music* 2/2 (1964), 56-76 and "Rational Structure in Music," *American Society of University Composers Proceedings* 11/12 (1976-77), 102-108. These and other theoretical writings are reprinted in *"Maximum Clarity" and Other Writings on Music* (Urbana, Illinois: University of Illinois Press, 2006) a volume of Johnston's collected writings edited by Bob Gilmore. Page numbers below refer to the versions in *"Maximum Clarity."*

conveying a complex musical relationships than tempered tuning. In particular, the hierarchical implications of just intonation lend a cognitive transparency lacking in symmetrical divisions of the octave like equal temperament:

> Interval scale thinking emphasizes symmetry of design. The harmonic and tonal meaning of symmetrical pitch structures is *ambiguity.* Chordally they produce either a sense of multiple root possibilities or of no satisfactory root possibility. Tonally they cause either a sense of several possible tonics or of no adequate tonic.
> Ratio-scale thinking, on the contrary, emphasizes a hierarchical subordination of details to the whole or to common reference points. The harmonic and tonal meaning of proportional pitch structures is clarity and a sense of direction.[32]

In the remainder of the article, Johnston derives a 53-tone scale based on five-limit just intonation, combined with a systematic use of accidentals which is the basis for his later notational practice (discussed in more detail below). This scale (with its careful accounting for syntonic commas using plus or minus signs before accidentals) was used compositionally in Johnston's String Quartet No. 2 (1964).[33]

Beginning with the solo trombone piece *One Man* (1967), Johnston began to extend his pitch language beyond 5-limit just intonation to embrace higher primes. *One Man* combined the septimal ratios of extended just intonation with the five-limit scales of his earlier work. The new resources of the seven-limit were explored in more detail in Johnston's String Quartet No. 4 (1973). This quartet, a variation form based on the American spiritual "Amazing Grace," progresses from Pythagorean intonation through

[32] Johnston, "Scalar Order as a Compositional Resource," 28.

[33] Von Gunden's monograph on Johnston's music includes a detailed analysis of this work (76-85). One of the most notable features is the structure of the first movement: each measure represents a step upward in Johnston's 53-tone scale. The cycle closes in the 54th measure, with a return to the original pitch level.

five-limit just intonation to septimal extended just intonation, as if recapitulating the history of tuning culminating with Johnston's system.[34]

These new musical resources were accompanied by a new theoretical tool, the harmonic lattice. Johnston credits this innovation to his acquaintance with writings by the Dutch theorist Adriaan Fokker (who advocated a 31-tone equal temperament to closely approximate just intonation including septimal intervals).[35] Lattices arrange pitches along axes corresponding to each of the prime numbers of a just intonational system; the number of axes could, in theory, be expanded indefinitely. Figure 3.9 reproduces Johnston's 3, 5, 7 lattice from his 1976 article "Rational Structure in Music."

[34] See Randall Shinn, "Ben Johnston's Fourth String Quartet." *Perspectives of New Music* 15/2 (Spring/Summer 1977), 145-73.

[35] Gilmore, Introduction to *Maximum Clarity,* xviii. For an introduction to Fokker's theories see A. D. Fokker, "Equal temperament and the thirty-one-keyed organ," *Scientific Monthly* 81 (1955): 161-166.

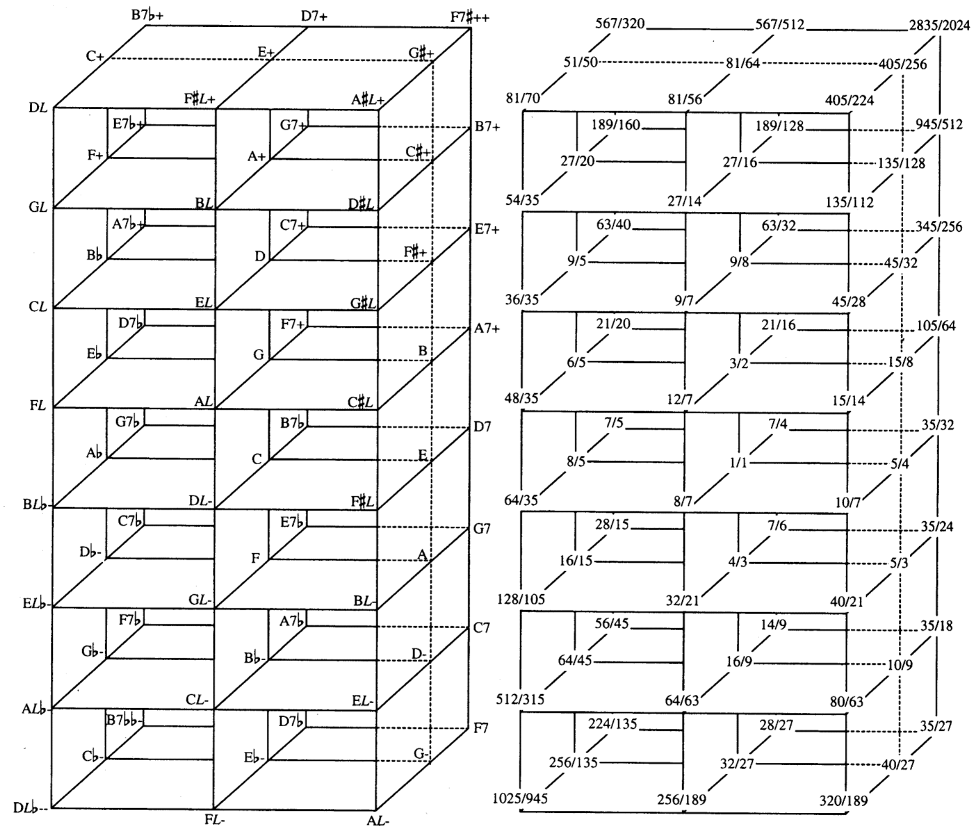

Figure 3.9: Johnston's 3, 5, 7 lattice[36]

The lattice appears in two different notations in this figure: the version on the left shows the note names of each pitch, while the version on the right shows the associated ratios. Like Partch, Johnston identifies each pitch by its relation to a central "tonic," 1/1. The left-right axis shows ascending just major thirds and the vertical axis ascending perfect fifths; Johnston's lattice adds a third dimension for natural sevenths (the ratio 7/4). The advantage of the lattice in conceptualizing and displaying intervals is that it concisely expresses our intuitions about distances in harmonic space: simply related pitches are near one another, while more complex ones are separated by a longer path through the nodes of the lattice. Johnston explains how scales in extended just intonation systems can be derived from a lattice: each scale begins with the "basic chord" of the

[36] Johnston, "Rational Structure in Music," 72.

system, comprised of 1/1 plus the nearest pitch on each axis, either in the "overtone" direction for a "major" system or the "undertone" direction for "minor." (This closely parallels Partch's Otonalities and Utonalities, and reflects Johnston's persistent interest in harmonic dualism and invertibility.) The scale of a given system expands on this "basic chord," subdividing each of its intervals (starting with the largest) by the addition of a new pitch that is *adjacent* in the lattice to the pitches already chosen.[37] A few additional rules ensure the compactness and regularity of scales based on any extended just intonation lattice. Writing in favor of lattices and ratio scales, Johnston again cites their comprehensibility: scales derived from ratios are "an effective aid in designing melodic and harmonic audible structure even with unfamiliar pitch materials."[38]

The note names in Figure 3.9 use some unfamiliar symbols from Johnston's microtonal notation system. Unlike Partch, who abandoned Western notation in favor of interval ratios treated as note names, Johnston works within the traditional system, but supplements it with new accidental symbols. His first step is to precisely define the tuning of each pitch on the staff when uninflected by an accidental. This white note scale is tuned to allow pure triads on F, G, and C, as in Figure 3.10.

C	D	E	F	G	A	B
1/1	9/8	5/4	4/3	3/2	5/3	15/8

Figure 3.10: Johnston's tuning of the "white-note collection"

This traditional tuning for the just major scale includes two different sized whole tones (9/8 between C and D and 10/9 between D and E), two different minor thirds (6/5 between E and G and 32/27 between D and F), and both perfect and out-of-tune fifths

[37] This process is closely related to James Tenney's theory of "crystal growth" in harmonic space, which is based on minimizing distances in harmonic space; this theory is discussed in detail in the section below on Tenney.

[38] Johnston, "Rational Structure in Music," 68.

(3/2 between C and G, 40/27 between D and A). The difference between each of these pairs of intervals is the syntonic comma, 81/80. Johnston uses the symbols + and – to indicate alterations of one syntonic comma: one multiplies by 81/80 to raise the pitch and by 80/81 to lower the pitch. Thus, the 9/8 ratio from C to D becomes a 10/9 ratio from C+ to D, and the out-of-tune fifth D-A can be made perfect by lowering D to D-. The adjustment by syntonic comma can be combined with other microtonal accidentals to describe any pitch on Johnston's lattices: Figure 3.11 lists Johnston's accidentals through the thirteen-limit.[39]

symbols		*ratio*	*cents*	*primary usage*
+	–	× 81/80	± 22 cents	raises 10/9 to 9/8, 32/27 to 6/5, 40/27 to 3/2
♯	♭	× 25/24	± 71 cents	raises 6/5 to 5/4
ㄥ	7	× 36/35	± 49 cents	lowers 9/5 to 7/4
↑	↓	× 33/32	± 53 cents	raises 4/3 to 11/8
13	ƐI	× 65/64	± 27 cents	raises 8/5 to 13/8

Figure 3.11: Johnston's accidentals, through the thirteen-limit

Normal chromatic adjustments are indicated by flats and sharps, expressly defined as multiplication or division by 25/24—the difference between the major third 5/4 and the minor third 6/5. For example, to lower B to B-flat, one multiplies the ratio of B, 15/8, by 24/25 with the result 9/5. For ratios in extended just intonation, Johnston uses accidentals which inflect pitches from within the five-limit. The "7" symbol can be combined with B-flat to lower the pitch from 9/5 to 7/4, the equivalent of multiplying by 35/36. To preserve the symmetry of the system, Johnston uses an inverted 7 to show a rise in pitch by the same interval. The arrows raise/lower the pitch by 53 cents, or 33/32: this is the difference between 4/3 (F) and the eleventh partial of C (11/4). The 13 and

[39] This figure is based on the table in Johnston's 2003 "A Notation System for Extended Just Intonation," in *Maximum Clarity,* 87. See also John Fonville's "Ben Johnston's Extended Just Intonation: A Guide for Interpreters," *Perspectives of New Music* 29/2 (1991): 106-137.

inverted 13 symbols work on the minor sixth—A-flat (8/5) is raised to 13/8 through multiplication by 65/64. Johnston uses the primes above 13 much less often, usually only in passages that treat the upper reaches of the overtone series as a sort of chromatic scale. In some sections of his String Quartet No. 9, the primes of 17, 19, 23, 29, and 31 appear in this context.

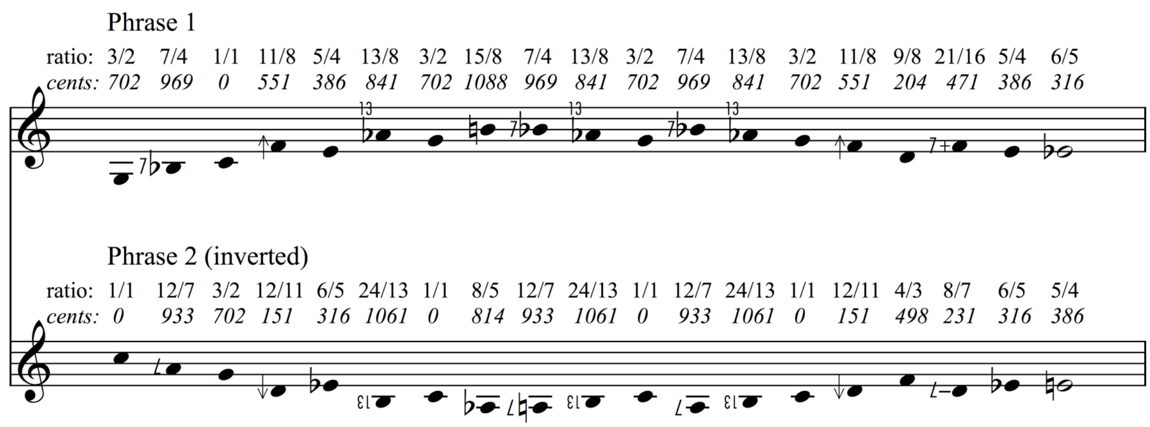

Figure 3.12: Opening of String Quartet No. 9, IV

Figure 3.12 diagrams the pitches of the opening two phrases of Johnston's String Quartet No. 9 (1987). The first phrase is based (with the exception of the final E-flat) on pitch classes drawn from the overtone series of C: this is recognizable from the presence of a power of 2 in each denominator. The second phrase is a precise inversion of the first, around the axis E/E-flat. As a result, the passage is built on "undertones" of G instead of overtones of C. The treatment of pitch material in these two phrases illustrates Johnston's long-standing fascination with invertibility and harmonic dualism.

Ezra Sims

The music of Ezra Sims (b. 1928) lies on the border between extended just intonation and the equal divisions of the octave proposed by microtonalists like Alois Hába and Julian Carrillo. Sims notates his work in a system dividing each semitone into six equal parts of 16 2/3 cents each; this results in a total of 72 pitches per octave.[40] His notation system is shown in Figure 3.13. The close spacing of pitches in 72-tone equal temperament makes it possible to closely approach all pitches within 8 1/3 cents; most of the partials from 1 to 33 can be approximated much more closely (see Figure 3.14).

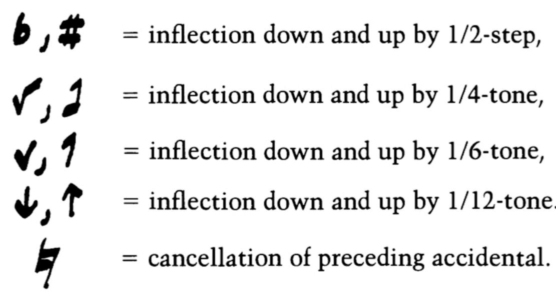

Figure 3.13: Sims's microtonal notation[41]

[40] As discussed later in this essay, 72-tone equal temperament has also been used by Georg Friedrich Haas and European composers; in the United States, saxophonist Joseph Maneri has used the 72-tone scale in both compositions and improvisations.

[41] Ezra Sims, "Yet Another 72-Noter," *Computer Music Journal* 12/4 (1988): 28.

partial	*cents*	*nearest 72-tone approximation*	*error*
33	53	50	–3
31	1145	1150	+5
29	1030	1033	+3
27	906	900	–6
25	773	767	–6
23	628	633	+5
21	471	467	–4
19	298	300	+2
17	105	100	–5
15	1088	1083	–5
13	841	833	–8
11	551	550	–1
9	204	200	–4
7	969	967	–2
5	386	383	–3
3	702	700	–2
1	0	0	0

Figure 3.14: approximations of partials 1 to 31 in 72-tone equal temperament

Though Sims notates his music in equal temperament, he expects that it will be heard as a close approximation of extended just intonation; ideally, players will also adjust pitches slightly to improve the fit with just intervals, much as they would when playing tonal music.[42] As does Johnston, Sims describes the mind's preference for just ratios in terms of a minimization of computational effort: "The mind will try to understand what it perceives according to its inbuilt biases—the preferences for verticals and horizontals, harmonic ratios, perhaps the Golden Section, and so on, that are apparently hardwired into it. If it can't, and is too long frustrated in the attempt, it's very likely to just say The Hell With It and go off and play with something else."[43] The mind's "hardwired" cognitive preference for rational intervals allows Sims's extended equal temperament to be heard as extended just intonation.

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[42] Ibid., 31.

[43] Ezra Sims, "Long Enough to Reach the Ground or How Long Should a Man's Legs Be?" *Perspectives of New Music* 32/1 (Winter 1994): 208-213: 211.

In his music, Sims uses an eighteen-note scale which can be transposed to various pitch levels within the 72-tone-per-octave tuning, much as the seven-note diatonic scale can appear at any transposition within the tempered twelve-tone chromatic scale. The equal temperament makes it possible to modulate from key to key without leaving the 72-note pitch world; modulation is even possible to such distant areas as the 11/8 augmented fourth or 13/8 "semimajor sixth."[44] Sims describes his scale (see Figure 3.15, which gives a just tuning for each pitch in relation to 1/1) as "an expanded diatonic": gaps in the standard diatonic scale are subdivided so that intervals between adjacent pitches range from quarter tones to third tones.

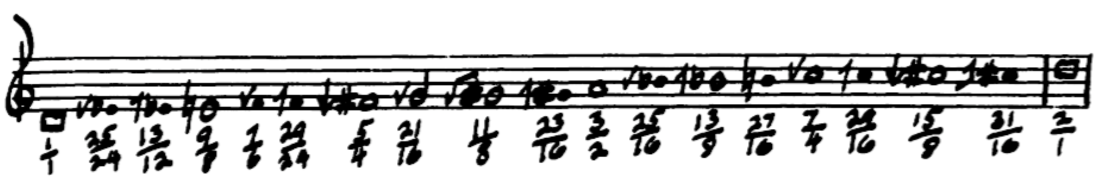

Figure 3.15: Sims's basic scale in the key of D[45]

For Sims, the scale and key relationships precede any particular harmonic technique: within the framework of his scale, Sims has used a variety of harmonic devices. One approach Sims describes is treating the scale degrees corresponding to the eighth to fifteenth partials as points of relative stability, which can be combined in various types of stable chords: "triadic, quartal, secundal, depending on the requirements of the piece."[46] Sims notes that pitches arranged to reflect difference and summation tone phenomena seem to reinforce one another particularly well: for instance, in the proportion 6:16:22, where 6 is the difference of 22 and 16, 16 is the difference of 22 and

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[44] Sims, "Yet Another 72-Noter," 31.

[45] Sims, "Reflections on This and That (Perhaps a Polemic)." *Perspectives of New Music* 29/1 (Winter 1991), 236-257: 241.

[46] Sims, "Yet Another 72-Noter," 39.

6, and 22 is the sum of 6 and 16. He speculates that such formations might "represent a sort of 'lowest energy state' requiring less effort of larynx and mind than would any collection of nearby but inharmonic pitches."[47] The combination of an interval with its sum and difference tones is closely related to the electronic techniques of ring and frequency modulation; applying these concepts to instrumental writing was common in the early works of the "spectralist" composers Grisey and Murail (and later Claude Vivier). Austrian microtonalist Franz Richter Herf frequently used arithmetic series (e.g. 1, 4, 7, 10...) with similar results.[48]

Sims argues that extended just intonation can also be found in jazz and blues music: he cites examples in specific performances by Odetta and Louis Armstrong. For Sims, these performances have essential microtonal components that are lost if the tune is transcribed into standard notation. In a transcription of Armstrong's "St. James Infirmary," Sims notes two different minor thirds, one a third tone smaller than the tempered minor third (7/6) and one a third tone larger (perhaps, Sims suggests, a 16/13 interval below 3/2: 39/32). In addition, Sims points out a melodic emphasis on two slightly different augmented fourths, approximating the ratios 11/8 and 23/16.[49] Sims uses this transcription as a source of melodic material in his *Sextet* (1983).

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[47] Ibid., 40-41. Murail describes the use of combination tones in his *13 Couleurs du Soleil Couchant* in "Target Practice," *Contemporary Music Review* 24/2-3 (2005): 165-166. Translation of "Questions de cible," *Revue Entretemps* 8 (1989).

[48] Horst-Peter Hesse, "Breaking into a New World of Sound: Reflections on the Ekmelic Music of the Austrian Composer Franz Richter Herf," *Perspectives of New Music* 29/1 (1991): 212-235. Horatiu Radulescu describes his own use of sum and difference tones in Bob Gilmore, "'Wild Ocean': An Interview with Horatiu Radulescu," *Contemporary Music Review* 22/1-2 (2003): 105-122. See also the discussion of Hans Zender's theories below.

[49] "Yet Another 72-Noter," 33-34.

Like Harrison and Johnston, Sims's aesthetic is closely tied to the musical textures and theoretical procedures of the classical tradition. Music by all three composers has recognizable scales, melodies, and counterpoint; the use of extended just intonation is considered an expansion, rather than a renunciation of the principles of traditional tonality. In their theoretical work, they build new musical languages by analogy to the languages of the past, although with a far richer range of intervallic and harmonic possibilities. A more radical branch of extended just intonation—influenced by Indian music, John Cage, and the avant-garde movements of the 1960s—can be seen in the work of La Monte Young and James Tenney.

La Monte Young

La Monte Young (b. 1935) found his way to extended just intonation independently of the research of Partch and his followers. For Young, the gateway to just intonation was his experience with Indian music, particularly as a disciple of the Indian vocalist Pandit Pran Nath. Young links the precise intonation of Indian *ragas* to the ratios of extended just intonation. The 3/7-lattice-based pitch organization of his six-hour improvised keyboard work, *The Well-Tuned Piano,* is discussed in detail in Chapter 1.

Other works include just ratios with higher prime limits. *The Melodic Version (1984) of The Second Dream of The High-Tension Line Stepdown Transformer from The Four Dreams of China* is a seventy-seven minute work for eight trumpets based on only four pitches, in the proportion 12:16:17:18; the sound is based on Young's childhood memories of the hum of electrical equipment near his Idaho home. The extraordinary length of these pieces is taken still further in his sine-wave installations—such as the extravagantly titled *The Base 9:7:4 Symmetry in Prime Time When Centered Above and Below the Lowest Term Primes in the Range of 288 to 224 with the Addition of 279 and 261 in Which the Half of The Symmetric Division Mapped Above and Including 288 Consists of the Powers of 2 Multiplied by the Primes Within the Ranges of 144 to 128, 72 to 64, and 36 to 32 Which Are Symmetrical to Those Primes in Lowest Terms in the Half of the Symmetric Division Mapped Below and Including 224 within the Ranges 126 to 112, 63 to 56, and 31.5 to 28 with the Addition of 119*—which sustain pure tones in complex just ratios for days at a time.[50]

Young is best known as one of the founders, along with Terry Riley, of musical minimalism, a term which seems problematic when applied to complex large-scale works like *The Well-Tuned Piano.* The description is more apt for Young's static, drone-based works, though these are still far removed from the repetitive minimalism of Philip Glass or Steve Reich (both members of a movement that can be traced back to Riley's *In C).* Like Young, Riley was a student of Pandit Pran Nath (the two also studied composition at the same time together at the University of California in Berkeley). Riley has also worked in just intonation—his works include pieces for retuned organs and pianos—*Shri Camel* (1980) and *Harp of New Albion* (1986) both use a five-limit just intonation.[51] Riley is less

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[50] Kyle Gann, "The Tingle of $p \times m^n -1$" in *Music Downtown: Writings from the Village Voice* (Berkeley: University of California Press, 2006): 269-271. See also Gann, "The Outer Edge of Consonance: Snapshots from the Evolution of La Monte Young's Tuning Installations" in *Sound and Light: La Monte Young and Marion Zazeela, Bucknell Review* XL/1 (Lewisburg: Bucknell University Press, 1996).

[51] See Kevin Holm-Hudson, "Just Intonation and Indian Aesthetic in Terry Riley's *The Harp of New Albion,"* http://www.ex-tempore.org/Volx1/hudson/hudson.htm, on the 5-limit just intonation tuning of Riley's 1984 piano cycle.

interested than Young in the new intervals of extended just intonation, and when retuning of instruments is impractical, Riley often plays on equal-tempered instruments.[52]

James Tenney

James Tenney (1934-2006) combined a strong grounding in physics and mathematics with an adventurous compositional spirit. Early in his career (in the mid-1960s), he worked on pioneering computer music projects at Bell Labs, while at the same time participating in New York's experimental and avant-garde art scene. With his interest in a broad range of disciplines, Tenney was able to bring a greater engagement with acoustics and psychoacoustics to bear on his theorizing than many of his compositional colleagues. The scientific side of Tenney's thought is clearest in the pragmatism of his musical theorizing—Tenney approaches the numerological abstractions of extended just intonation with a skeptical eye, always aware of how musical structures are actually perceived.

Perhaps the most far-reaching of Tenney's contributions to just intonation theory is the idea of tolerance. This is implicit in the work of other composers (Ezra Sims's tempered just intonation, for example) but Tenney provides a more thoroughly considered treatment of the subject. For Tenney, tolerance is "the idea that there is a certain finite region around a point on the pitch height axis within which some slight mistuning is possible without altering the harmonic identity of an interval."[53] A just

[52] See Riley's interview with Frank Oteri on the website *New Music Box*, "Terry Riley: Obsessed and Passionate About All Music," http://www.newmusicbox.org/article.nmbx?id=1288.

[53] James Tenney, "The Several Dimensions of Pitch." In *The Ratio Book: A Documentation of the Ratio Symposium, Royal Conservatory, The Hague, 14-16 December 1992,* ed. Clarence Barlow (Cologne: Feedback Studio Verlag. 102-115): 109.

interval with its particular harmonic quality can tolerate a degree of mistuning before that quality is lost. The degree of tolerance is variable, depending on many contextual factors, but in general varies "inversely with the ratio complexity of the interval." Simple intervals like octaves and fifths would retain their identity under greater mistuning than complex intervals like 9/8 or 17/16. A result of tolerance is that there is a practical limit on the complexity of a just intonation system; very complex ratios are likely to be heard as out-of-tune variants of simpler ratios. With the addition of tolerance, the abstractions of just intonation become applicable to a wide range of pitch phenomena—just intonation plus tolerance can explain tempered and inharmonic sonorities as well as those based on pure harmonic structures.

Tenney's idea of harmonic simplicity is formalized in a notion of *harmonic space.*[54] As in Ben Johnston's harmonic lattices, Tenney's harmonic space consists of a number of discrete points joined by axes representing prime integers. The number of axes can be theoretically infinite, but Tenney's concept of tolerance constrains this proliferation because complex intervals built of high prime factors tend to be heard as mistuned simple ratios. One of Tenney's innovations is the addition of a weighted metric for harmonic distance: see pages 125-128 for a detailed discussion of Tenney's formula for calculating harmonic distance.

These distance measurements allow Tenney to be very specific about how pitch sets are perceived harmonically: we understand the set as the closest possible arrangement in harmonic space. In his article "On 'Crystal Growth' in Harmonic Space,"

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[54] James Tenney, "John Cage and the Theory of Harmony." In *Soundings 13: The Music of James Tenney.* Santa Fe, New Mexico: Soundings Press, 1984. 55–83. Also in *Musicworks* 27 (1984), 13–17. Reprinted in *Writings about John Cage.* Ed. Richard Kostelanetz. Ann Arbor, Michigan: University of Michigan Press, 1993. 136–61.

Tenney explores how pitch sets might develop through the addition of pitches one by one, always adding the point in pitch space that minimizes the sum of harmonic distances between all pitches in the set. This approach gradually builds pitch-space crystals growing from Pythagorean sets based only on 2 and 3 to five-limit just intonation sets and eventually extended just intonation structures like the crystal in Figure 3.16.[55]

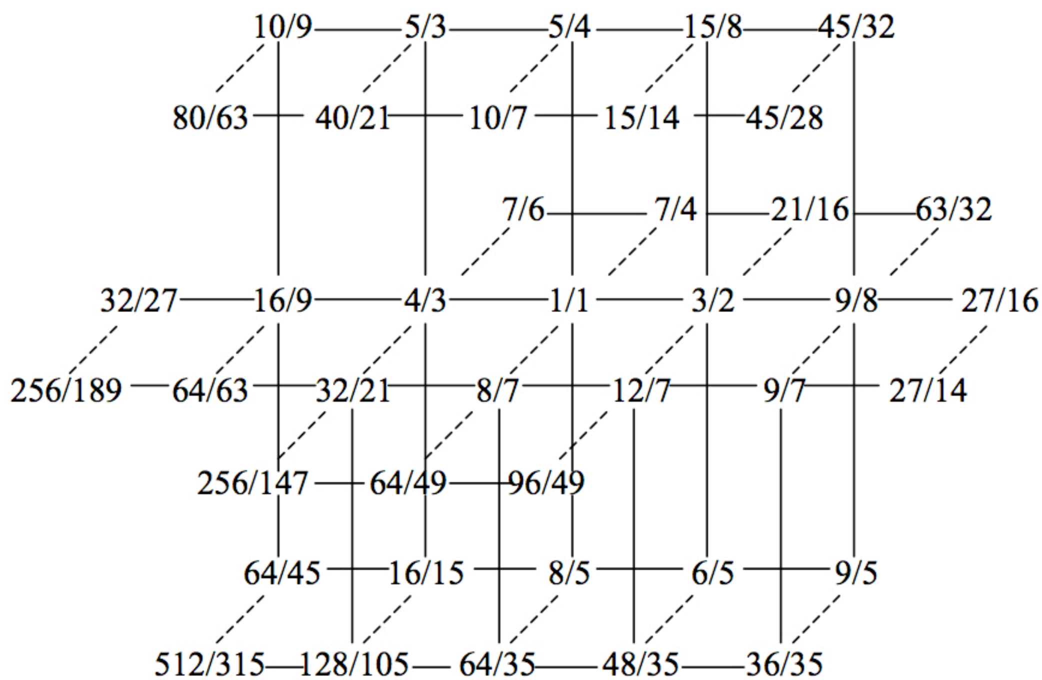

Figure 3.16: A "harmonic crystal" in 3,5,7-space

Throughout his compositional career, Tenney has realized just intonation in a number of different ways: unlike the "purist" system-building composers Partch and Johnston, Tenney is willing to employ different methods of pitch organization, even including temperament. One of these methods is 72-tone equal temperament (as in the work of Ezra Sims). In *Changes: 64 Studies for Six Harps* (1985), this temperament is realized by the retuning of six harps, each to a different pitch level; the staggered equal-

[55] "Ausweitung in eine neue Dimension: 'Kristallwachstum' im harmonischen Tonraum (1993-1998)." *MusikTexte* 112 (February 2007): 75-79. English version, "On 'Crystal Growth' in Harmonic Space (1993-1998)" in *Contemporary Music Review* 27/1 (2008): 47-56.

tempered chromatic scales of the harps combine to produce 72 equal steps. Tenney has also used up or down arrows (or combinations of arrows) to indicate the division of the semitone into six parts (or even seven parts as in *Glissade* (1982), for a more precisely approximated fifth partial. In early works like *Clang* (1972), down arrows were used to (imprecisely) show the lowering of the 7th and 11th partials. Tenney's preferred notation for recent works has been "cents deviation": this shows the deviation of the desired pitch from the nearest tempered note in cents. Although he recognizes that this degree of precision may be impossible to attain, Tenney feels this notation best conveys his desired pitch relationships: "it's a small target, but it's still a target, right?"[56] This compromise with tempered tuning would have been anathema to a just intonation purist like Partch, but it offers greater accessibility for performers accustomed to tempered pitch notation.

Other American Just Intonation Composers

While the figures discussed above may be the best known American composers working in extended just intonation, many other composers have engaged aspects of just intonation theory through their compositions.

Like James Tenney, Alvin Lucier (b. 1931) often bases his compositions on acoustic phenomena. Though Lucier does not explicitly use just intonation in his works, they address many of the same psychoacoustical phenomena that are essential to just intonation theory—from beats between closely spaced tones to the spectral structure of individual instrumental tones. One of his best known pieces, *I Am Sitting in a Room,* makes audible the resonant frequencies of its performance space. The performer records a

[56] James Tenney and Donnacha Dennehy, "Interview with James Tenney," *Contemporary Music Review* 27/1 (2008), 79-90: 80.

brief text ("I am sitting in a room, different from the one you are in now..."), which explains the concept of the piece in detail. The recording is then played in the same room, where it is recorded and then replayed: this process of recording and playback is repeated over and over again. As the process is repeated, certain frequencies of the recording are amplified by the room's natural acoustical resonances, while others are cancelled out. In the final cycles, only haunting whistles remain: "We discover that each room has its own sets of resonant frequencies in the same way that musical sounds have overtones."[57] Many of Lucier's works combine acoustic instruments with electronically generated sine tones: in *Music for Piano with Slow Sweep Pure Wave Oscillators* (1992), Lucier sets the piano's sounds against sine waves which change slowly in pitch. As the sine waves cross the instrumental sounds, they set off complex patterns of interference and beating: the sine waves set the spectral structure of the piano tones into relief, making hidden acoustic phenomena audible.

Pauline Oliveros (b. 1932) occupies an important place among composers exploring just intonation as part of a spiritual, meditative practice. Her music—often presented in verbal scores as "Sonic Meditations"—combines breathing and meditation exercises with improvisation (Oliveros performs on a justly-tuned accordion). Through these meditation, Oliveros pursues an intensely aware state of musicality she calls "deep listening."[58]

A new generation of just intonation composers has emerged, including students of Ben Johnston (Kyle Gann) and James Tenney (Larry Polansky and John Luther Adams).

[57] Alvin Lucier, "Careful listening is more important than making sounds happen," in *Reflections: Interviews, Scores, Writings* (Cologne: MusikTexte, 1995), 434.

[58] Heidi Von Gunden. *The Music of Pauline Oliveros* (Metuchen, New Jersey: Scarecrow Press, 1983).

Just intonation has informed works in a variety of styles, ranging from Ellen Fullman's delicate works for her "Long String Instrument" (with justly tuned strings from thirteen to thirty meters long) to Glenn Branca's high-volume "symphonies" for multiple electric guitars (with other instruments including electronic keyboards and drums) tuned to pitches of the harmonic series.[59] The intricacies of tuning theory have been pursued by a number of theorists, often working outside the mainstream academic establishment: these theorists include John Chalmers, Ervin Wilson, and Ivor Darreg, as well as the members of an active online discussion group.[60]

Extended Just Intonation in European Composition

Extended just intonation has been a largely American phenomenon, due largely to the lasting influence of Partch and Johnston. In the 1950s and 60s, just tuning seemed to particularly thrive in the Midwest and West, areas less closely associated with European musical thought than the urban centers of the East Coast, where serialism prevailed in the increasingly academic world of new music. While microtonality was certainly not new in European new music, European composers tended to explore expanded equal temperaments rather than extended just intonation—see for example the music of Alois Hába or Ivan Wyschnegradsky. Although the subdivision of the octave into twenty-four or more equal parts can be used to approximate extended just intervals, in most cases composers using such divisions have been more interested in exploring their mathematical and combinatorial properties, using the finer division of the octave to

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[59] Ellen Fullman, "The Long String Instrument," *Musicworks* 85 (Spring 2003): 20-28. Glenn Branca's interview with William Duckworth appears in *Talking Music* (New York: Schirmer, 1995).

[60] http://groups.yahoo.com/group/tuning, accessed April 15, 2008.

support new patterns and scales.[61] Ideas from extended just intonation occasionally emerge in the work of composers using expanded equal temperament—for example, Wyschnegradsky describes the interval of five and a half semitones as related to the 11/8 ratio. But in general, such ratio considerations are subservient to an overriding model based on the abstract geometries of pitch distance.[62]

Karlheinz Stockhausen's *Stimmung* (1968) was one of the most influential works to adopt the intervals from the harmonic series associated with extended just intonation. The entire work for six vocalists is based on a single overtone series, built on a low B-flat and including the seventh and ninth partials. The singers' vowel sounds are carefully combined to emphasize different combinations of formants—the result is closely related to the Tuvan and Tibetan practices of overtone singing. The work is steeped (like the music of La Monte Young) in Eastern mysticism: in addition to the collaborative, ceremonial treatment of the vocal ensemble, the text of the work consists of the "magic names" from a variety of cultures around the world and short texts written by Stockhausen.

Among composers of Stockhausen's influential generation, György Ligeti has been most interested in exploring the world of extended just intonation, though he uses this

[61] In his article "Mikrotonalitäten," Georg Friedrich Haas suggests that there are four distinct approaches to microtonal pitch organization: "1) tempered divisions of the octave into equal parts other than twelve; 2) tuning systems based on the proportions of the overtone series (just intonation); 3) 'tone-splitting' (Klangspaltung), that is, the use of very small intervals close to the unison: the focus of compositional interest is the beating and interference between the tones; and 4) aleatoric microtonality through particular instrumental techniques, whose pitch is not precisely specified: for example, prepared piano, some percussion instrument sounds, the ad-libitum detuning of strings, etc." Georg Friedrich Haas, "Mikrotonalitäten," in *Musik der anderen Tradition: Mikrotonale Tonwelten* (Munich: Musik-Konzepte, Edition Text+Kritik, 2003), 59-65.

[62] A possible counterexample are the composers writing in 31-tone equal temperament associated with Adriaan Fokker in the Netherlands; we have already seen the influence of Fokker's theories on Ben Johnston's multidimensional tone lattices. The 31-tone division of the octave allows close approximations of seven-limit just intervals. Perhaps because of the difficulties of realizing this music without specially constructed keyboard instruments, the tuning has not found widespread acceptance beyond a small circle.

material with a personal stamp. Ligeti's works have used a variety of means to create a kind of "hybrid microtonality," combining different kinds of pitch theorizing including just intonation, temperament, and uncertain or wavering tuning. These include retunings of parts of the ensemble (as in *Ramifications* and the *Violin Concerto*), microtonal inflections (the *Double Concerto*), the use of high harmonics on strings and brass (the *Cello Concerto* and *Hamburgisches Konzert*), and even historical temperaments (*Passacaglia ungherese)*. Ligeti has referred to some of the pitch adjustments in the *Double Concerto* as "Partch effects"; another link to the American extended just intonation composers is the influence of Ligeti's pupil Manfred Stahnke, who studied with Ben Johnston in Illinois. Like Ligeti, Stahnke often combines aspects of just intonation with other intonational systems for an intentionally impure hybrid, as in his 1987 *Partch Harp* for retuned harp and synthesizer.[63]

The questions addressed by composers working in extended just intonation have also fascinated composers of the spectral movement. Tristan Murail and Gérard Grisey are the best known of this group, and widely regarded as the movement's founders, but spectralism has influenced many other composers including Kaija Saariaho, Magnus Lindberg, and Jonathan Harvey.[64] The differences between spectralism and just intonation are subtle but significant. While just intonation composers tend to base their theories around pure tunings and frequency ratios, spectral composers ground their work in the analysis of actual sonorities—often including inharmonic or otherwise distorted

[63] Bob Gilmore, "The Climate Since Harry Partch," 29-30. Stahnke describes how just intervals are used in several of his works in "Mein Weg zu Mikrotönen," in *Musik der anderen Tradition: Mikrotonale Tonwelten* (Munich: Musik-Konzepte, Edition Text+Kritik, 2003), 125-140.

[64] One could argue that Olivier Messiaen anticipated many of the concerns of the spectralists; see Julian Anderson's "A Provisional History of Spectral Music," *Contemporary Music Review* 19/2 (2000): 7-23.

spectra. Thus, it is not uncommon to see spectral composers using pitch sets based on the stretched spectra of piano strings—sometimes this natural stretching is even exaggerated for a stronger effect. The just intonation composers tend to refer to the pure harmonic spectrum as a kind of Platonic ideal for harmonic relationships—those who are more scientifically inclined locate the organizing power of the ideal overtone series in the mechanisms of aural perception. Spectral composers, on the other hand, locate the "natural" in the acoustical structure of actual sounds: they are thus much more aware of distortions from ideal harmonicity together with the actual amplitude of each spectral component and its temporal evolution. This aspect of spectral thought has been encouraged in recent years by the availability of computer applications for sound analysis. Spectralists and extended just intonation composers tend to call for very different degrees of intonational precision: while extended just intonation calls for precisely defined pitches (or at most, no more than about 8 cents of error, as in Ezra Sims's 72-tone temperament), spectralism tends to approximate pitch information to a quartertone or eighth-tone grid. Perhaps most importantly, spectralism carries a different aesthetic history—one far closer to European serialist and post-serialist music. The different aesthetics are clear if one compares the traditionalist textures of Harrison or Johnson with spectral music—the Indian-based music of Young or Tenney's minimalist structures are also distant from spectralism's more active and dramatic musical discourse.[65]

Tempered forms of extended just intonation (in the manner of Ezra Sims) have been explored independently by a number of European composers. The Austrian

[65] Among spectral composers, Horatiu Radulescu's largely static and precisely tuned works (for example, *Inner Time II* (1993) for 7 clarinets) show the greatest similarity to the music of Young and Tenney. See Gilmore, "'Wild Ocean': An Interview with Horatiu Radulescu."

composer Franz Richter Herf (1920-1989) designed his own system of microtonal notation dividing the octave into 72 equal parts; his system was inspired by the microtones of Croatian folk music and the theories and music of Alois Hába. Herf was particularly interested in collections of pitches whose partial numbers form an arithmetic series: that is, a series in the form a, a+b, a+2b, a+3b, etc. A combination of such series provide the pitch organization for his *Ekmelischer Gesang* (1975) for solo violin.[66] Georg Friedrich Haas (b. 1953) has also adopted 72-tone equal temperament: his works in this tuning system include the seventy-five-minute orchestral work *in vain* (2000), which combines overtone-based harmonies with twelve-tone equal temperament tritones, fourths, and fifths. His First String Quartet (1991) retunes each string of the quartet to allow complex overtone-based combinations of natural harmonics, much in the manner of La Monte Young's *Chronos Kristalla* (1990).[67] Hans Zender has also turned to 72-tone equal temperament in several works, drawn by its combination of close approximations of the upper partials and a capacity for lightning-quick modulations and harmonic multivalence. Zender is particularly interested in the harmonic effects of sum and difference tones (as created by the electronic technique of ring modulation—see Stockhausen's *Mantra* among many other works). Among the complex structures he derives by this procedure is the elegant Fibonacci-series harmony formed by the pitches approximating overtones 1, 2, 3, 5, 8, 13, 21, 34, 55, etc., where each new upper pitch is

[66] Horst Peter Hesse, *Grundlagen der Harmonik in mikrotonaler Musik.* (Innsbruck: Edition Helbling, 1989) and "Breaking into a New World of Sound," *Perspectives of New Music* 29/1 (Winter 1991): 212-235.

[67] Haas, op. cit., 64-65.

the sum of the harmonic numbers of the two pitches below. In such a series, the intervals between adjacent pitches gradually converge on the same size, about 833 cents.[68]

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The many facets of extended just intonation discussed here underscore the powerful appeal of this approach to composers of the mid- to late-twentieth century. By engaging the emerging scientific discipline of psychoacoustics, extended just intonation has become a promising, scientifically supported theory, not just an archaic remnant of a tonal past. As Harry Partch declared, "It need hardly be labored that music is a physical art, and that a periodic groping into the physical, a reaching for an understanding of the physical, is the only basic procedure, the only way a music era will attain any significance."[69] In its promise of a consistent system built on acoustic facts, just intonation is able to fulfill many of the desiderata of a modernist music theory; in particular, an independence from the languages of the past (as represented by equal temperament) and the possibility for limitless expansion into the outer reaches of the overtone series. Since the early experiments of Harry Partch, extended just intonation has developed from a fringe phenomenon to a more central position in the discourse around pitch structure in new music.

Historically, the emergence of just intonation must be viewed in part as a reaction to other musical developments. James Tenney has described the appeal of extended just intonation as a way of expanding beyond the "exhausted harmonic resources" of the

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[68] Hans Zender, "Gegenstrebige Harmonik" in *Musik der anderen Tradition: Mikrotonale Tonwelten* (Munich: Musik-Konzepte, Edition Text+Kritik, 2003), 167-208. Also in *Die Sinne Denken: Texte zur Musik 1975-2003*, ed. Jörn Peter Hiekel (Wiesbaden: Breitkopf & Härtel, 2004). Péter Eötvös also explored the Fibonacci series intervals in his *Intervalles-Intérieurs* (1974): see his liner notes to *Intervalles-Intérieurs, Windsequenzen,* Budapest Music Center Records, BMC CD 092, 2003.

[69] Harry Partch, "Show Horses in the Concert Ring," in *Bitter Music,* 174-180.

standard twelve-tone collection. Extended just intonation allowed the construction of complex harmonies and musical surfaces without abandoning tonality—it suggests an expanded tonality rather than atonality. For other composers, just intonation was a liberation from the abstractions of serial music. Like spectral music, music in extended just intonation could abandon the motivic and geometric play of serial structure in favor of simple, transparent forms—forms built explicitly to engage the mechanisms of our aural perception.

Extended just intonation represents a turn away from structural listening—listening syntactically to the development of themes, motives, and architectonic forms—and toward a meditative, in-the-moment mode of listening. The appeal of just harmonies over tempered harmonies is that the just harmonies are defined *in their own terms,* not designed as part of a system of syntactic relationships. That is, our appreciation of just harmonies focuses on their nature, not their place in a system of relationships. As Stockhausen said of *Stimmung:* "Time is suspended. One listens to the interior of the sound, to the interior of the *harmonic spectrum,* to the interior of a vowel: TO THE INTERIOR"[70] This turn from music based on narrative forms combining many sounds to the contemplation of a single sound (with, of course, its own complex internal overtone structure) appears in many guises: Pauline Oliveros's "deep listening," Lucier and Young's sound installations, Tenney's meditative *koans* and swells, and the "one note" orchestra works of Giacinto Scelsi. The psychedelic aesthetic of the 1960s and the simultaneous explosion of interest in Eastern spirituality were important influences on the spread of "deep listening," as were John Cage's exhortations to explore the nature of the

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[70] Karlheinz Stockhausen, liner notes to *Stimmung: Pariser Version,* Stockhausen Edition, CD 12, 1993: 72.

sounds that surround us. As long as composers continue to be inspired by the physical and sensual properties of musical tones and their combinations, extended just intonation will play an important role in compositional theory.

CHAPTER 4: Gérard Grisey and the "Nature" of Harmony

Introduction: Spectral music and the idea of the "natural"

Gérard Grisey (1946–1998) was a founding member of the spectralist movement—a group of French composers born in the 1940s whose best-known members are Grisey, Tristan Murail, Michaël Levinas, and Hugues Dufourt. Spectralism emerged in the 1970s, in part as a reaction against the abstraction of serial music. Instead of basing their music on the manipulation of rows or motives, spectral composers take inspiration from the physical properties of sound itself. Each composer's interpretation of what "spectral music" might mean is slightly different, but as a generalization we could say that its essential characteristic is the dissection of sounds into collections of partials or overtones as a major compositional and conceptual device. Spectral composers use the acoustical fingerprints of sounds—their spectra—as basic musical material.

In their writings, spectral composers often emphasize the *natural* origins of this material. In this chapter I explore how Grisey's music invokes the idea of nature, and what this idea might mean for listeners and analysts. I will suggest that the word "nature" can have two contrasting interpretations in Grisey's music—one based on the objective, physical nature of external reality, the other on the subjective, internal nature of aural perception—and that these two interpretations of "nature" lead to very different ways of thinking about and analyzing his works.

As discussed in Chapter 1, one of the most characteristic procedures of spectral composition is "instrumental synthesis": this technique mimics the electronic music technique of additive synthesis, but replaces pure sine tones with the complex sounds of real instruments. To review the technique, we return to the iconic example from the beginning of Grisey's *Partiels* for chamber orchestra (1975), already discussed briefly in Chapter 1. The opening of the work is based on a *fortissimo* trombone E_2. The trombone sound can be analyzed into a set of partials of varying frequencies and amplitudes—this can be expressed as a numerical table, or graphically as a spectrogram: see Figure 4.1.[1] The composer then scores these partials for an instrumental ensemble. Figure 4.2 summarizes the opening page of the score—we first hear the trombone (accompanied by *sforzandi* in the double bass)—as the trombone fades out, instruments from the ensemble gradually enter, playing pitches which match the analyzed partials of the trombone sound. This process is repeated several times. Grisey's goal is not a precise reproduction of the trombone sound—which would be impossible with the complex spectra of acoustic instruments—but rather a hybrid sonority in which we can hear both the individual instruments and their fusion into a unified timbre.

[1] Figures 4.1 and 4.2 reproduce Figures 1.25 and 1.26 from Chapter One.

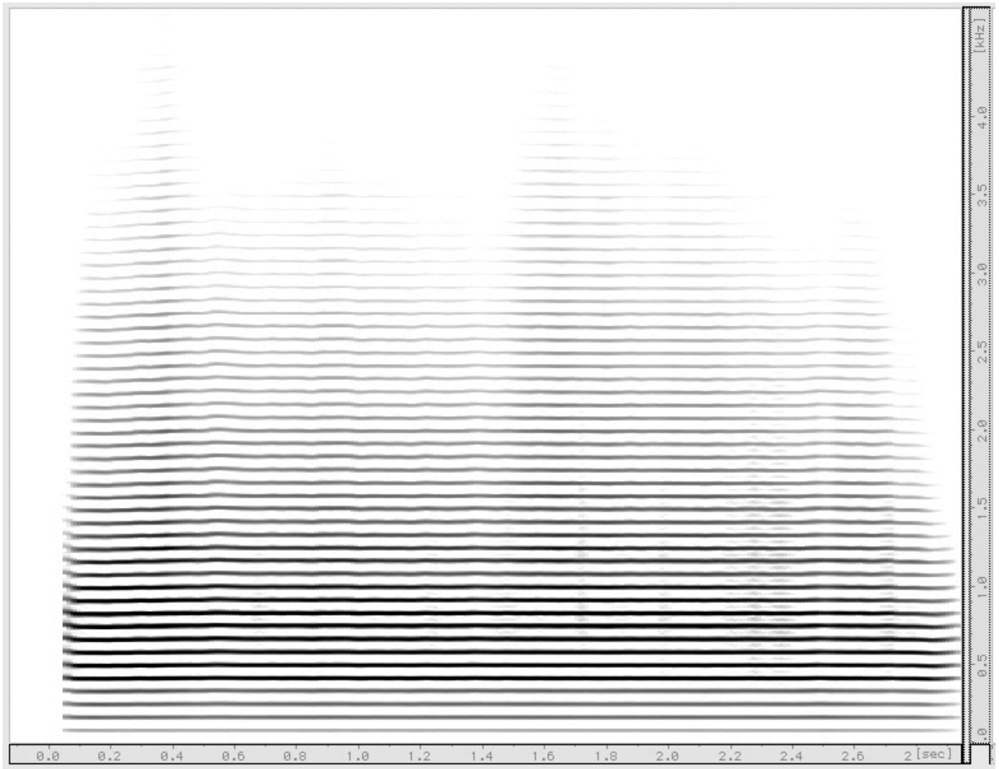

Figure 4.1: Spectral analysis of a trombone sound

Figure 4.2: Instrumental synthesis of a trombone sound in Grisey's *Partiels* (1975)

The physicality of sound is brought into focus by these techniques of analysis and resynthesis—this is an appeal to nature in the external, objective sense. For music theorists, Grisey's technique will have strong echoes of Rameau's *corps sonore*—the essential difference, though, is that Grisey is dealing with real sounds, not an idealized source of overtones. Recall that Rameau's *corps sonore,* as formulated in the *Génération Harmonique*, conveniently stopped vibrating after the sixth partial to avoid the "out-of-tune" natural seventh.[2] In contrast, Grisey carries the complexities of real sounds into his music, including their often distorted and imperfect spectra.

Some of the most common musical sounds have *inharmonic* spectra. A piano string, for example, produces a stretched spectrum: only an idealized string with no mass or stiffness would produce a pure harmonic spectrum. This means that the first overtone, at the octave, is not exactly twice the frequency of the fundamental, but rather slightly higher. The stretching continues into the higher partials—we might not realize it (though our piano tuners do), but by the fourth octave the partials of a low piano note are almost a third of a tone (65 cents) above their equivalents in a pure harmonic series. Other spectra are compressed, like those of certain brass instruments: each partial is lower than its harmonic counterpart.

Grisey acknowledges these real-world departures from ideal harmonicity in the design of his 1996 chamber ensemble piece *Vortex Temporum.* His compositional procedures are extensively documented in sketch-study monographs by Hervé and Baillet, and sketches in the Paul Sacher Stiftung confirm their findings.[3] In Figure 4.3, I retrace some of Grisey's techniques to illustrate how he brings inharmonic spectra into his music. His conceptual point of departure is a naturally stretched sound—in this example, I illustrate this with the piano's lowest B-flat. The first staff of Figure 4.3 shows a pure harmonic spectrum on B-flat; the second shows the pitches of the partials of a real piano B-flat, computed by Fourier analysis. Each partial is indicated by the nearest equal-

[2] See Thomas Christensen, *Rameau and Musical Thought in the Enlightenment* (Cambridge: Cambridge University Press, 1993): 133-168.

[3] See Jean-Luc Hervé, *Dans le vertige de la durée:* Vortex Temporum *de Gérard Grisey* (Paris: L'Itinéraire, 2001) and Jérôme Baillet, *Gérard Grisey: Fondements d'une écriture* (Paris: L'Harmattan, 2000).

temperament pitch, with the deviation from equal temperament supplied in cents beneath each note.

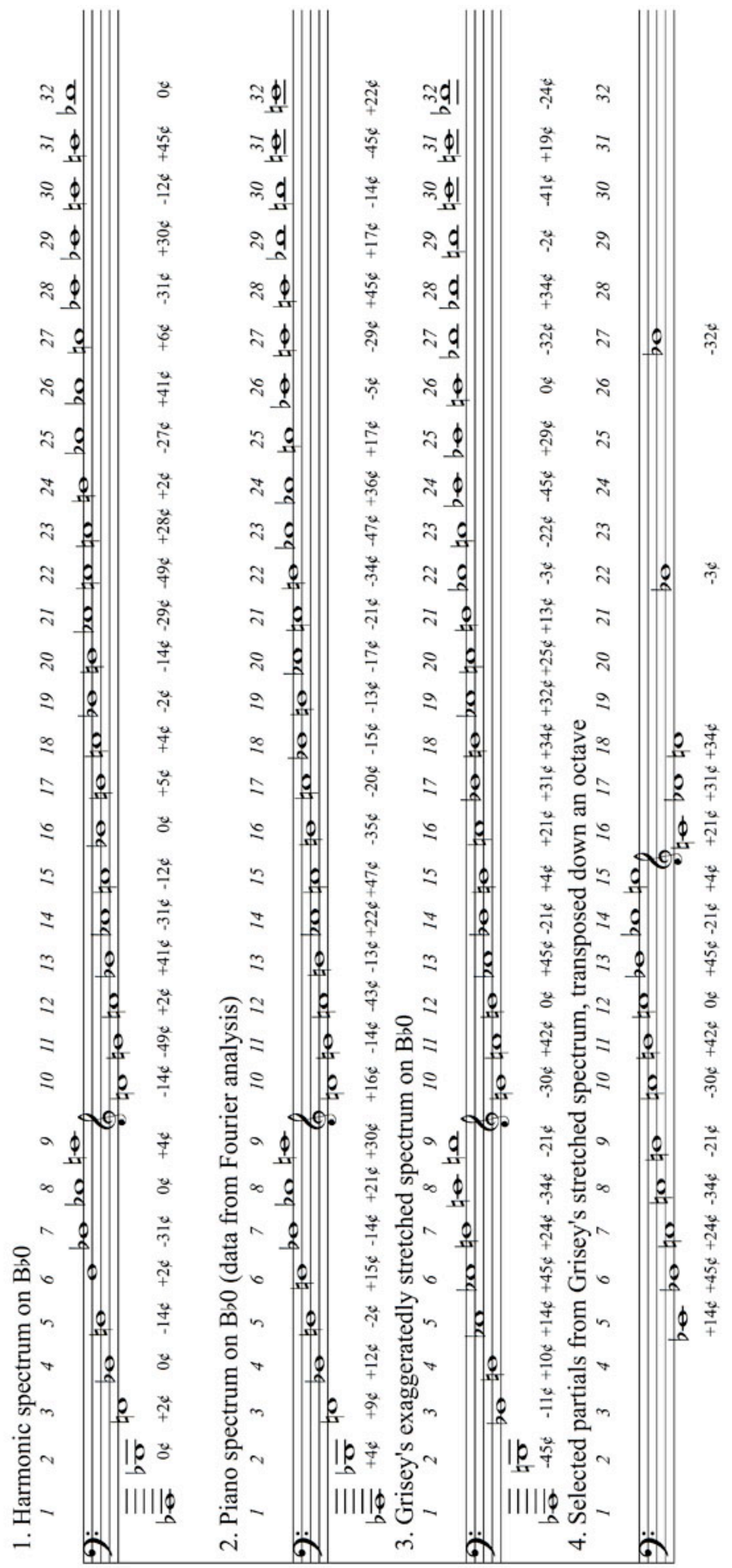

Figure 4.3: Four spectra. Deviations from equal temperament indicated in cents under each note.

Grisey is not content with the relatively tame degree of stretching of the piano note, and exaggerates it until each octave is expanded by a quartertone—see the third staff in Figure 4.3.[4] While the sixteenth partial of the piano tone is 65 cents sharp, in Grisey's spectrum it is 221 cents sharp—more than a whole tone above the equivalent harmonic partial. Grisey's distorted spectrum is based on a property of natural sounds—the stretching found in piano spectra—but exaggerates that property to an unnatural degree. Why does Grisey exaggerate the stretching so drastically? In part, it may be to make the spectrum's inharmonicity apparent even when the partials are rounded off to a quartertone grid—with a smaller degree of stretching, the approximation to quartertones would erase the difference between stretched and harmonic spectra. Also, the exaggerated distortion produces a sonic result reminiscent of the broken octaves characteristic of much serial music—despite spectralism's professed antipathy to serialism, the characteristic sound of serial music continued to exert a strong influence on spectral composers. Most importantly, though, I think we must consider Manfred Stahnke's assertion that Grisey is primarily interested in thresholds and borderline cases: in this case, the perceptual threshold where, as the degree of stretching increases, a spectrum is no longer heard as a fused timbre but instead as a collection of independent pitches.[5]

In *Vortex Temporum,* the spectrum is removed yet further from its natural model by "filtering" (his term for the omission of certain partials) and transposition down by an

[4] Grisey also chooses a different curve to describe the spectral distortion. The equation for the frequency of partials of a piano string is $fn = nf_0(1+Bn^2)^{1/2}$, where n is the partial number, f_0 is the fundamental frequency, and B is a constant defined by the physical properties of the string. Grisey opts for the simpler equation: $fn = f_0 n^{1.046}$. See Jérôme Baillet, op. cit., 217 and Harvey Fletcher, E. Donnell Blackham, and Richard Stratton, "Quality of Piano Tones," *Journal of the Acoustical Society of America* 34/6 (June 1962), 756.

[5] See Manfred Stahnke, "Die Schwelle des Hörens: 'Liminales' Denken in Vortex Temporum von Gerard Grisey" *Österreichische Musikzeitschrift* 54/6 (June 1999): 21-30.

octave. Figure 4.4 shows the three types of spectra (compressed, harmonic, and stretched) used in *Vortex Temporum.*

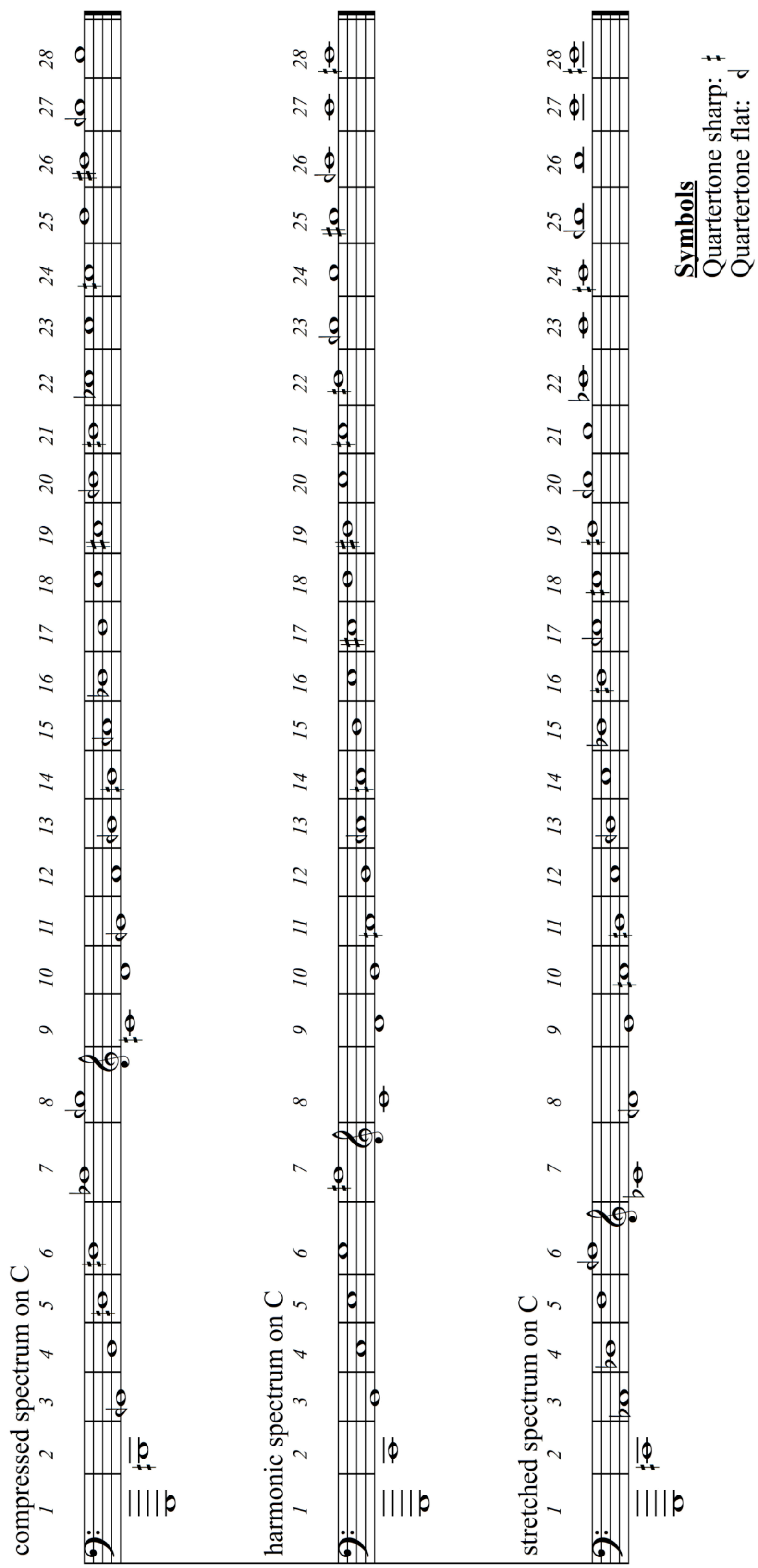

Figure 4.4: three spectrum types used in *Vortex Temporum*

This is how “objective nature” enters Grisey’s music: as the reproduction through instrumental synthesis of the acoustical spectra of real-world sounds, with their characteristic distortions kept intact and even emphasized. Straightforward as this evocation of nature might seem in an early work like *Partiels,* when we listen to later works like *Vortex Temporum* it’s often impossible to hear Grisey’s harmonies as versions of the natural spectra from which they were derived. After the harmonies have been subjected to extensive compositional manipulation—exaggerated stretching or compression, approximation to a quartertone grid, and the omission of many partials, often including the fundamental—their natural acoustical source is no longer recognizable.[6] What, then, does talking about Grisey’s compositional techniques really tell us about how we hear his music? When the “natural” derivation of his material collapses, and we hear the distorted spectra as complex chords rather than fused timbres, the appeal to nature in the objective, external sense fails, and the workings of our internal nature—the nature of our auditory perception—become more relevant to our musical understanding.

* * *

Instead of analyzing the music by reconstructing Grisey’s derivation of the harmonies, we can use the theory of tone representation to approach the music beginning from our own harmonic intuitions.[7] This is a turn from one sense of the “natural” to the

[6] Significantly, the organizational power of harmonicity is lost when we change from using harmonic to inharmonic spectra. One of the unique qualities of harmonic sounds is that they are easily resolved into objects when presented at the same time. But, as Bregman notes (op. cit., 238) “when two stretched series of partials are sounded at the same time, you do not hear only two distinct sounds as you do when listening to two harmonic sounds.”

[7] Manfred Stahnke takes a similar approach in his analysis of chords from the first part of *Vortex Temporum.* See Stahnke, “Zwei Blumen der reinen Stimmung im 20. Jahrhundert: Harry Partch und Gérard

other: from an external idea of the natural, based on how Grisey's harmonies draw on natural models, to an *internal* idea of the natural, based on how we intuitively—that is, naturally—make sense of complex sonorities. We can describe Grisey's harmonies without reference to their source, but rather starting with our own aural experience.

Composer Roger Reynolds has questioned the relation between spectral models and their musical realization:

> An interesting but problematic aspect of some contemporary French thinking about musical practice is the reference to avowedly scientific sources for modeling the acoustic behaviors of sounds and thereby establishing a rationale for redefining, for example, the relation between timbre and harmony. An instance of the difficulty that one encounters in considering their perspectives can be directly stated: an examination of the spectral profile of a particular sound over time does, indeed, allow one to identify the components of its idiosyncratic perceptual impact upon the ear. It is, however, clearly absurd to hold that a "by hand" representation (approximation) of such spectral structures on the basis not of sinusoidal partials, but rather by means of instrumental tones (each of which has its own, nonconforming, nonintegral spectrum), could possibly result in an orchestrated product that bears anything other than an incoherent and metaphoric relationship to the supposed model. The fact that this rationale is not, in fact, rational, may not necessarily invalidate its artistic effects, of course; but, if one does not hold the analogy to have validity, the appeal is arbitrary.[8]

Reynolds is at pains to emphasize that his characterization of the relationship between spectral structures and their acoustical models as "incoherent and metaphoric" is not an aesthetic judgment so much as a theoretical one. Reynolds's observations raise an

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Grisey." *Hamburger Jahrbuch für Musikwissenschaft* 17 (2000): 369-389 and "Die Schwelle des Hörens" (op. cit.). Manfred Stahnke characterizes Grisey not as a "spectral" composer, but instead a "liminal" composer, interested in experimenting with the edges of our "shape-finding" abilities. As Albert Bregman notes, inharmonic spectra can often induce ambiguous or weak pitch perceptions; playing with this ambiguity is essential to Grisey's aesthetic. In support of this view, Stahnke points out how Grisey uses his three spectral types as pools for characteristic material rather than attempts to recreate specific acoustic sonorities. Even the "just" spectrum is tempered, and often uses different pitch classes for 7th and 14th partials—see Stahnke, "Die Schwelle des Horens," 22.

8 Roger Reynolds, "Seeking Centers," *Perspectives of New Music* 32/2 (1993): 282-83. This critique is similar in tone to criticisms of serialism as perceptually opaque: see Fred Lerdahl's "Cognitive Constraints," op. cit.

important question: if the models underlying the music are not in a clear, unambiguous relationship to the musical surface, then does an analysis based on a reconstruction of compositional procedure tell us anything about how the music is heard and understood? If the model is not clearly reflected in the musical surface, an analysis from the perceptual standpoint is likely to tell us more about the work. Tone representation is a useful tool for modeling our perceptual intuitions about harmony in spectral works when the underlying acoustic models cannot satisfactorily explain how harmonies are heard.[9] By applying tone representation, we can approach these works through the analytical lens of extended just intonation.

Vortex Temporum I

Vortex Temporum, a three-movement work for flute, clarinet, violin, viola, cello, and piano, poses some intriguing analytical difficulties. The work is a "spectral" composition, insofar as it is composed by reference to models based on the acoustic spectra of instruments. However, Grisey's compositional techniques significantly alter these spectra, sometimes to the point of unrecognizability.[10]

As noted above, the compositional techniques and plans which Grisey used to construct the work have been described in detail in studies based on the composer's sketches for the work. The description of compositional process, however, is not necessarily a good description of a piece's aural and musical effect. Even though many

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[9] Note that in one sense, the process of tone representation is the reverse of the spectralist procedure of instrumental synthesis. Instrumental synthesis approximates natural just-tuned overtones with tempered pitches, while tone representation can take a tempered approximation and restore its just intonation implications.

[10] Portions of this section are adapted from my article, "Tone Representation and Just Intervals in Contemporary Music" *Contemporary Music Review* 25/3 (June 2006), 263-281.

spectral techniques take acoustic and psychoacoustic facts as their starting point, there is often no clear, unambiguous relationship between such compositional techniques and their audible musical results.

Manfred Stahnke has made some brief but tantalizing observations about how some of the harmonies of *Vortex Temporum* might be interpreted by ear—his interpretations address aspects of the harmonies which are not evident from a consideration of their compositional origin (see note 6, above). By examining the beginning of the work's first movement through the lens of tone representation, I hope to offer some new insights into its harmonic relationships as I hear them. Tone representation can function as a sort of "listening grammar" for complex microtonal sonorities.[11]

* * *

The first two minutes of the first movement use a very limited set of harmonic materials—we hear the alternation of three distinct "chords," arpeggiated by the flute, clarinet, and piano.[12] Each of these three chords was conceived by Grisey as a subset of a "stretched" harmonic series. The normal harmonic spectrum is systematically distorted, so that each octave is stretched to approximately an octave plus a quartertone. From the resulting distorted spectra, Grisey selects certain pitches for each chord. Figure 4.5, drawing on the sketch-based research of Baillet and Hervé, illustrates the derivation of the three chords (labeled, in order of appearance, *x, y,* and *z*).

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[11] The term "listening grammar" was coined by Fred Lerdahl in his article "Cognitive Constraints on Compositional Systems," in *Generative Processes in Music: The Psychology of Performance, Improvisation, and Composition,* ed. John Sloboda (Oxford: Oxford University Press, 1988): 231-59. Lerdahl makes a distinction between listening grammars (which a listener uses to make sense of a musical work) and compositional grammars (which a composer uses to create a work).

[12] The arpeggiated figure is intentionally derived from a gesture in Ravel's *Daphnis and Chloe*. See Hervé and Baillet (op. cit.) for details.

Figure 4.5: Derivation of Vortex Temporum chords x, y, and z from stretched spectra.

As discussed above, the stretched harmonic spectrum is conceived as an analogy to the mildly inharmonic spectra of many common musical sounds. If Grisey's more intensely stretched spectra were played in their entirety, and with simple sine tone partials, we might recognize their connections to their acoustic models. However, in *Vortex Temporum,* only a small subset of each spectrum is heard, and that is "resynthesized" using complex instrumental timbres. The selection of a subset of pitches from a spectrum can completely efface the spectral derivation of that subset—careful selection of pitches can even imply that their source is a different spectrum altogether. Grisey's own awareness of this possibility is clear from a sketch in the Sacher Stiftung—by selecting carefully from the stretched G spectrum in Figure 4.5, he derives subsets which imply a stretched D-sharp spectrum (D-sharp/D-¾sharp/B/E/G-sharp—partials 3, 6, 9, 12, and 15) and even a compressed F-sharp spectrum (F-sharp/F-¼sharp/C-¼sharp/F-¼sharp—partials 7, 13, 19, and 25). Given this gap between spectral compositional procedures and the aural effect of the derived chords, recounting the compositional process sheds little light on the way the harmonies are actually heard. It will be more productive to analyze the chords without reference to their derivation, concentrating instead on how we might hear the chords: their tone representations, internal tensions, and relationships to one another.

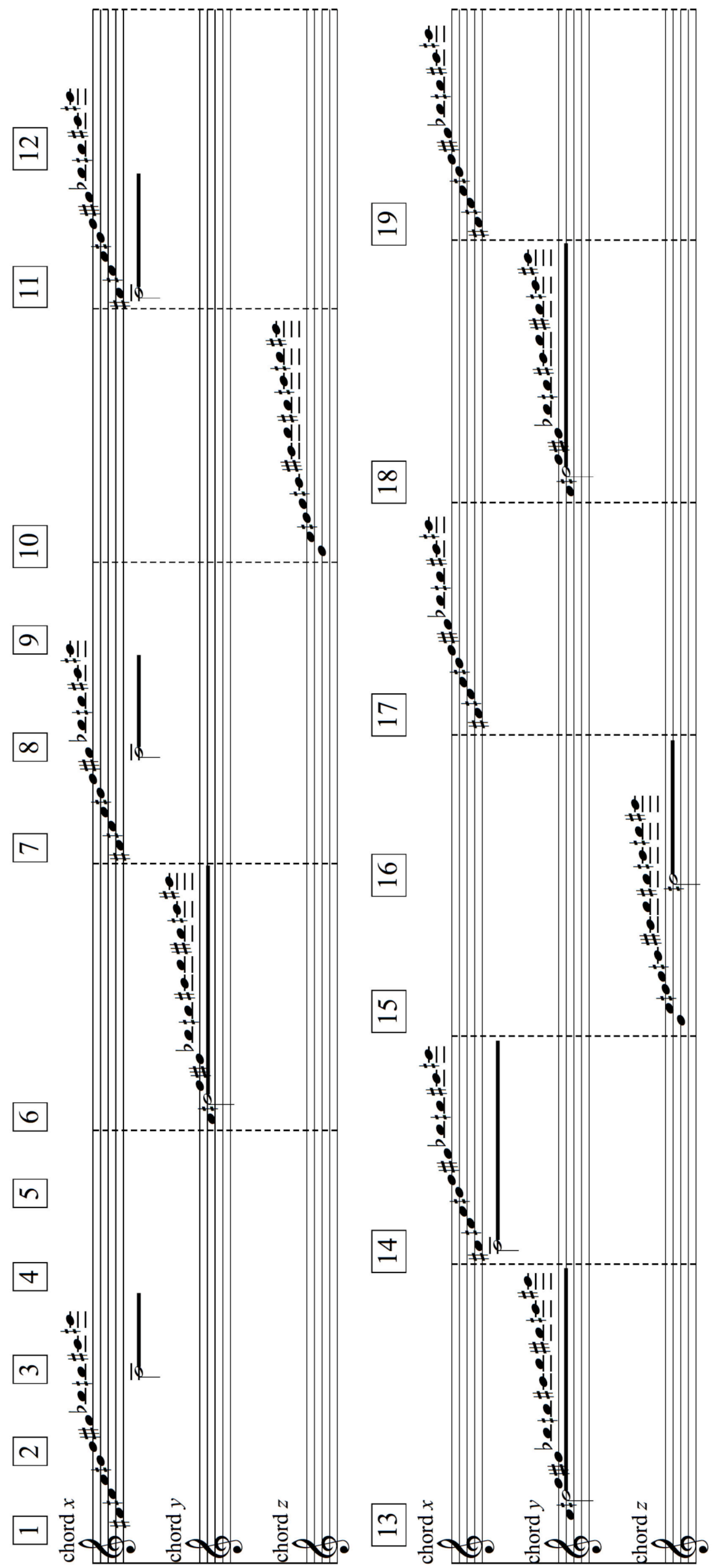

Figure 4.6: Arrangement of chords x, y, and z in Vortex Temporum, rehearsal numbers 1 to 19.

Figure 4.6 shows how the three chords *x, y,* and *z* are deployed in rehearsal numbers 1 through 19 (this figure is based loosely on Baillet, p. 214). At each rehearsal number, the arpeggiation is punctuated by a cluster of very high piano notes, and sometimes quick notes in the strings—however, these short-lived punctuations do not seem to substantially affect the harmonic perception of the three sustained arpeggio chords, so they will be omitted from this analysis. Sometimes, the arpeggiation is joined by sustained single tones in one of the strings (shown here as half notes followed by bars indicating their length). As fixed points opposing the rapidly moving arpeggios, these sustained tones capture our aural attention very strongly, and they will be given corresponding weight in our analysis.

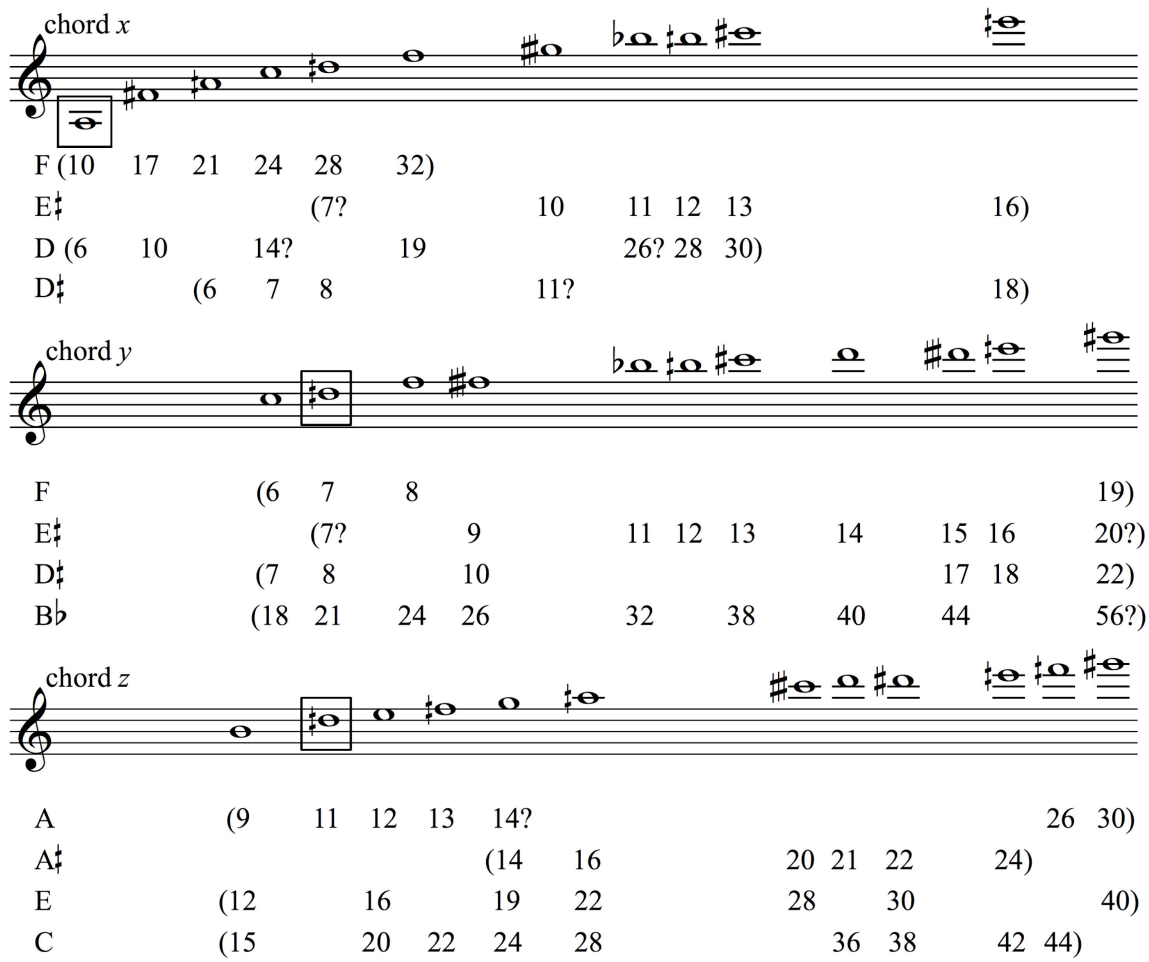

Figure 4.7: Tone representation analysis of Vortex Temporum chords x, y, and z.

chord *x*

Figure 4.7 illustrates some possible tone representations for the chords *x, y,* and *z.* (Boxed pitches indicate the notes which are sometimes held as "pedal points" by the strings.) At the beginning of *Vortex Temporum*, the arpeggiation of chord *x* is sustained for more than thirty seconds; it seems false to musical experience to assert that we don't begin making sense of this chord until there's something to compare it to. One advantage of tone representation is that we can say things about collections of pitches without comparing them to other collections, as in pitch-class set analysis. The technique of pitch-class set analysis is essentially comparative; in technical terms, this means that we can say very little about a pitch collection until we find something to compare it to. In many musical situations, however, we will want to say something right away. The idea of tone representation will allow us to make observations about the internal composition of this complex chord from the very beginning of the work. This analytical approach does not view the identification of motivic repetition within a work as the most significant type of relationship—instead, we can cite the harmonic tendencies of any given collection, along with any internal tensions or ambiguities of tone representation. These tensions are essential in making the chord compelling to listen to for such a long time.

One way of hearing the first chord is as a combination of tones related to two harmonic centers, D and D-¼sharp (see Figure 4.7): the A and F-sharp imply a D root, while the A-¼sharp and D-¼sharp imply D-¼sharp. This tone representation is essentially what we would expect from the compositional derivation of the chords: the stretching process moves the C-sharp "fundamental" up to D, then to D-¼sharp. However, the pitches G-¾sharp and B-flat don't fit well with either the D or D-¼sharp fundamentals (the question marks on Figure 4.7 indicate that the corresponding pitches are unusually out of tune for the indicated tone representation). Describing the chord as based on D or D-¼sharp (or, for that matter, as a stretched C-sharp spectrum) gives only an overall, statistical impression, which doesn't account well for details; to me, though, the harmonic details are precisely what makes this harmony interesting.

Another, more nuanced interpretation recognizes an F root for the lower part of the sonority, combined with an E-¼sharp root for the upper part. An advantage of this interpretation over the D/D-¼sharp one is that we can recognize a definite tone representation for every pitch, accepting less mistuning between the sounding pitches and their just intonation tone representations. Aurally, I find this tone representation much more convincing, even though one must accept quite high partial classes, 17 and 21, to account for the F-sharp and A-¼sharp as part of the F spectrum. A low F played beneath the harmony seems to fit well as a "root" for the lower half of the chord; this is largely due to the harmonic strength of the fourth C/F (an interval which was problematic in the D/D-¼sharp reading). This interpretation gives us a clear explanation for the prominent C-F fourth, and does not require that we distort it into a more complex interval.

The arpeggio figuration tends to temporally separate the upper and lower parts of the chord. Playing the upper part of the chord alone (from the G-¾sharp up) makes its orientation toward E-¼sharp clearly audible. By recognizing a clearly delimited E-¼sharp collection above the F collection, the sonority seems much more like a *compressed* spectrum than a stretched one (as the chord's derivation would suggest); we also have a good explanation for the quartertone-flat octaves F/E-¼sharp and C/B-¼sharp, which are responsible for much of the chord's characteristic tension.

The ambiguity between the two possible tone representations of chord *x*—as a stretched spectrum with D and D-¼sharp roots or a compressed spectrum on F and E-¼sharp—seems to reflect an essential part of Grisey's musical style. As Stahnke writes, "Grisey plays with the shape-finding capability *(Gestaltfindungsfähigkeit)* of our ear, with the thresholds of our awareness."[13] The play between two plausible tone representations helps to animate the chord through this long passage.[14]

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[13] Stahnke, "Zwei Blumen," 383.

[14] In reference to the opening harmony of *Vortex Temporum I* (based on a stretched C# spectrum), Manfred Stahnke writes: "We hear a certain quasitonal harmonicity, but can't find any "tonic feeling" for the tone

chord *y*

The first major harmonic change in the piece occurs at rehearsal number 6, with the move to chord *y.* As we know, this collection shares a common origin with chord *x*; both are subsets of the same stretched spectrum. We can easily hear that the two chords are closely related, since more than half of chord *y*'s pitches are common tones with chord *x*. Does this mean that the chords have the same harmonic effect? In fact, the changed pitches, and particularly the overall higher register, make the new chord's tone representations subtly different from those of chord *x*.

We hear a continuation of the F/E-¼sharp tone representation quite strongly—indeed, the F root of the lower part of the chord is much stronger than in chord *x,* in the absence of the F-sharp and A-¼sharp which were fairly weak partial classes of F. Without these pitches, the C/F fourth can be represented as 6:8 rather than the more complex 24:32. Three of the four new pitches (F-¾sharp, D, and D-¾sharp) fit well with the E-¼sharp fundamental, while the high G-sharp could be F(19) or a rather flat E-¼sharp(20).

The sense that the chord might be heard as based on D is considerably weakened by the absence of the low A and F-sharp. However, we can still sense the possibility of D-¼sharp as a root for three of the lower notes of the chord (C, D-¼sharp, and F-¾sharp, but not F-natural). This is a byproduct of the stretched spectrum in the compositional process—if we take a sample from further up in the spectrum, the higher the perceived "root" of that sample will be. The sense of D-¼sharp as fundamental is strengthened by

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low C#, since Grisey begins with the fifth partial tone and through his spectral distortion reaches an extreme polyvalence of harmony: the neutral third (F# to A quartersharp) could also suggest a *compressed* spectrum on a "fundamental" F#: see also the diminished fifth C over F# and the natural seventh compressed to a sixth D-quartersharp." Stahnke, "Die Schwelle des Horens," 22.

the absence of the A and F-sharp, which were "out-of-tune" with the D-¼sharp root, and by the added F-¾sharp, which is D-¼sharp (10).

It's also possible to recognize a weaker tone representation based on B-flat. The strength of the B-flat interpretation is that it can relate nearly all of the pitches to a single fundamental, but the very high partial numbers that it requires make it less convincing than the F/E-¼sharp interpretation.

When the music returns to chord *x* from chord *y* at rehearsal number 7, our perception of chord *x* is subtly colored by the experience of chord *y.* The sustained D-¼sharp, a sustained note in the viola in chord *y,* is heard as closely linked to the sustained cello A in chord *x.* The strength of this linkage makes an "in tune" version of the interval A/D-¼sharp desirable—since this interval is clearly emerging as a harmonically important one, it makes sense to understand it as a representation of a just interval. Only the F tone representation allows us to hear the interval between A and D-¼sharp as a just relationship in the context of chord *x,* F(10:28), further strengthening the sense of an F root as opposed to D or D-¼sharp.

chord *z*

At rehearsal number 10, we hear chord *z* for the first time. The pitches of this chord were selected from a spectral source set a tritone lower than that of chords *x* and *y.* Is there any way that we experience this harmonic change as a "tritone transposition"? Many of the pitches of chord *z* are a tritone below pitches in chords *x* and *y* (F, G-¾sharp, B-flat, B-¼sharp, and C-sharp in chord *x* map onto B, D-¼sharp, E, F-¼sharp, and G in chord *z*). However, the overall range remains the same, which works against the sense of tritone transposition; also, as Stahnke points out, the chords are linked by close voice leading, creating a sense of continuity with chord *x* and weakening the sense of tritone transposition. "In comparison to the spectrum on C-sharp, we experience an almost stationary *(gleichbleibendes)* tone field, that seems only to be illuminated by a different light. The quartertone movements are important for this effect."[15]

Even acknowledging the many pitches in chord *z* which are related by a tritone to pitches in chord *x,* I don't hear an overall transposition by tritone here. To me, the most aurally convincing account of the sonority is as a combination of pitches implying an A root in the lower part of the chord and an A-¼sharp root in the upper part. (The sense that the sustained cello A of chord *x* lingers aurally into chord *z* strengthens this tone representation.) While in chords *x* and *y,* the upper part of the chord was based on a fundamental a quartertone below the fundamental of the lower part (E-¼sharp above F, or in spectral lingo, a "compressed" spectrum), here, the upper part of the chord is based on a fundamental which is a quartertone *above* that of the lower part (A-¼sharp above A, or a "stretched" spectrum). The contrast between these two types of tone representations further weakens the case for a tritone transposition—we have a sense of a different type of harmony, not a transposition of the same type. If we do hear a change of overall "root" from chord *x* to chord *z,* it is likely to be from F to A—transposition up a third, not down a tritone.

It's also possible to hear tone representations of E and C in this sonority—particularly before the entrance at rehearsal number 16 of the sustained viola D-¼sharp, which contradicts both of these tone representations. The D-¼sharp strongly focuses our tone representations toward the A root for the lower notes of the chord; the lowest

[15] Stahnke, "Zwei Blumen," 382.

trichord, B/D-¼sharp/E, is only possible with A as fundamental: A(9:11:12). The common D-¼sharp sustained tone with chord *y* also makes chord *z* sound similar to *y* in some ways—both are heard as contrasts to the more ubiquitous chord *x.* The shared high G-sharp and nearly identical overall register also help to cement a relationship between the chords, although they are harmonically quite different.

The return of the D-¼sharp over chord *z* as a sustained tone is a striking harmonic event; this sustained pitch has previously only appeared with chord *y,* and we hear it quite differently in its new context. D-¼sharp is now heard as A(11) instead of F(7), and the dyad of sustained string notes A/D-¼sharp is reinterpreted to A(4:11) from F(10:28). The reinterpretation of this emphasized dyad, highlighted by the long duration of its component notes, can be heard as an essential part of the harmonic move from F to A.[16]

Vortex Temporum II

Figure 4.8 is an outline of Grisey's deployment of spectra in the second movement of *Vortex Temporum*. Our analysis will begin with the stretched B-flat spectrum in section II of the movement (rehearsal numbers 4 to 7), schematized in Figure 4.9. All the pitches of the spectrum are rounded off to the nearest quartertone. The pianist cycles continuously downward through the boxed notes on the bottom staff (four pitches of the piano are retuned a quartertone flat to allow the performance of microtonal intervals). The pianist re-articulates the stemmed notes on every beat. The other instruments of the ensemble—flute, clarinet, violin, viola, and cello—play sustained tones: these are shown on the upper two staves of Figure 4.9 with bars indicating their duration.

[16] One can hear a similar harmonic relationship, but in reverse, in the excerpt from Ligeti's *Melodien* discussed in Chapter 2*:* the dyad B/E-flat changes from B(16:22) to G(10:14).

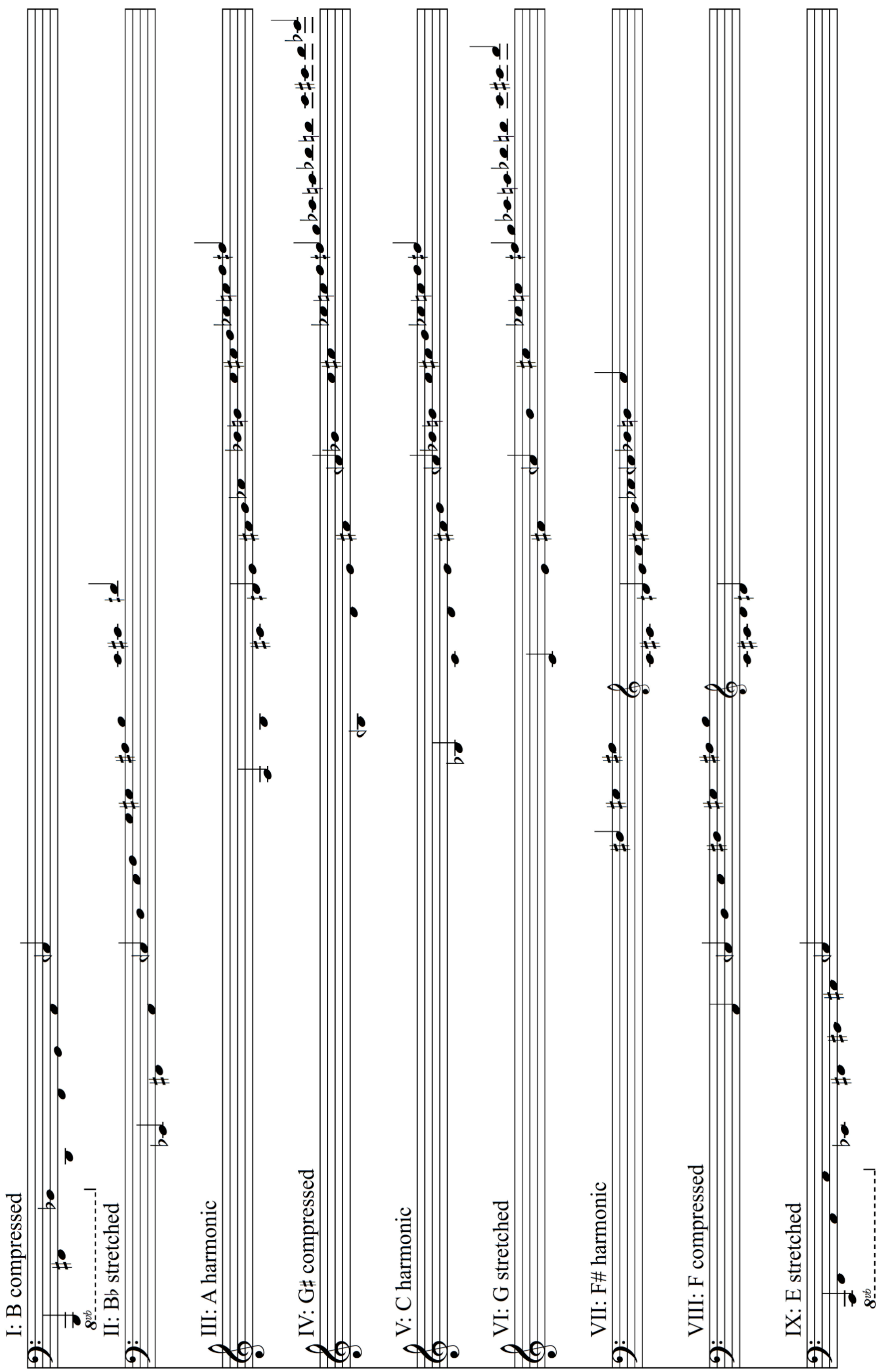

Figure 4.8: Spectra in *Vortex Temporum II*

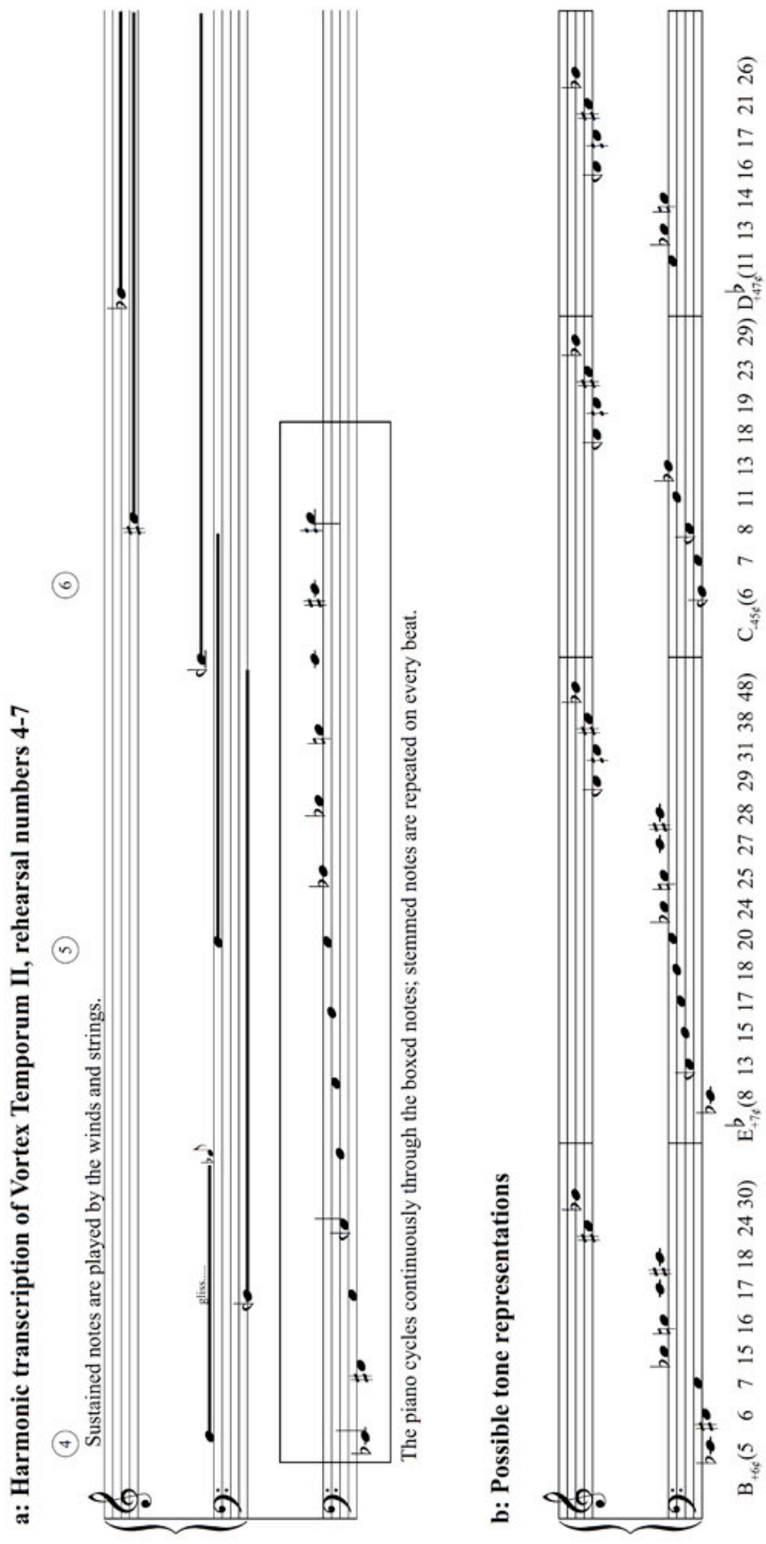

Figure 4.9
a: Harmonic transcription of *Vortex Temporum II,* rehearsal numbers 4–7
b: Possible tone representations

The descending pitches of the piano imitate the well-known aural illusion of the endlessly descending Shepard tone—the descent seems continuous because as the entire complex of partials drifts downward, new high partials gradually fade in from silence and the lowest partials drop out: see Figure 4.10.[17]

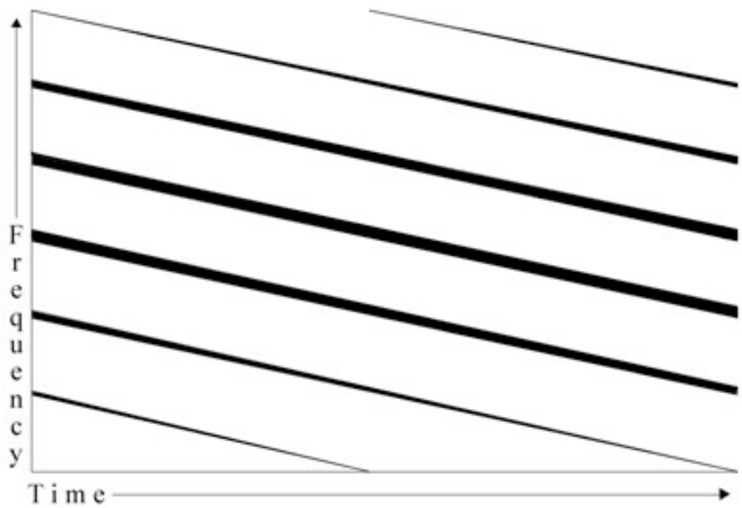

Figure 4.10: schematic illustration of a descending Shepard-Risset glissando

Rather than explain the harmonies of *Vortex Temporum* by reference to Grisey's compositional derivation of the chords from natural models, we will focus on the experience of listening, taking tone representation as a model for our natural harmonic intuitions. Since no one fundamental offers a convincing tone representation for all the pitches in the excerpt, in Figure 4.9b I've identified four plausible tone representations, each describing different subsets of the complete harmony. Our choice of one tone representation over another depends on the musical context, and changes over the course of the passage. Often, not all pitches are equally important to our decision—we give more weight to repeated and held notes. The sonority rooted on B seems especially convincing

[17] See Roger Shepard, "Circularity in Judgments of Relative Pitch," *Journal of the Acoustical Society of America* 36 (1964): 2346–2353. An example of a continuously descending Shepard tone can be found on the website of the Acoustical Society of America: http://asa.aip.org/demo27.html (accessed April 15, 2008).

at the beginning of the excerpt, when the B is present as a held note in the flute and violin. However, the B representation cannot account for some salient pitches in the texture: the held G-¼flat and the repeated C-¼flat and D-¼sharp. As the B fades out, our attention turns to the tone representations which give these salient pitches more weight, and we experience the sonority as a combination of harmonies with roots on E-flat and C-¼flat (the low repeated notes in the piano). At the end of the excerpt, the upper-register held pitches G, D-¼flat, F-sharp, and Bb strongly imply the tone representation D-¼flat(11:16:21:26)—the piano's B-flat, B, and D-¼sharp also fit into this interpretation, as partials 13, 14, and 17.

Figure 4.11 shows two analyses of the last spectral chord of the movement. Analysis 1 follows Grisey's derivation of the chord, describing it as the third through twelfth partials of a stretched spectrum on E. From a perceptual standpoint, though, this interpretation is full of problems—for example, we're asked to hear the lowest four notes, C-F-A-C as implying E rather than F as a fundamental. My adaptation of Riemann's tone representation, linking pitches by the simplest possible just intervals, leads to the much more satisfying Analysis 2. The chord consists of pitches drawn from two distinct harmonic roots: the notes C-F-A-C-D# are the third through seventh partials of an F fundamental, while the remaining F#-G#-A#-C-¼flat are the eighth through eleventh partials of an F# fundamental.

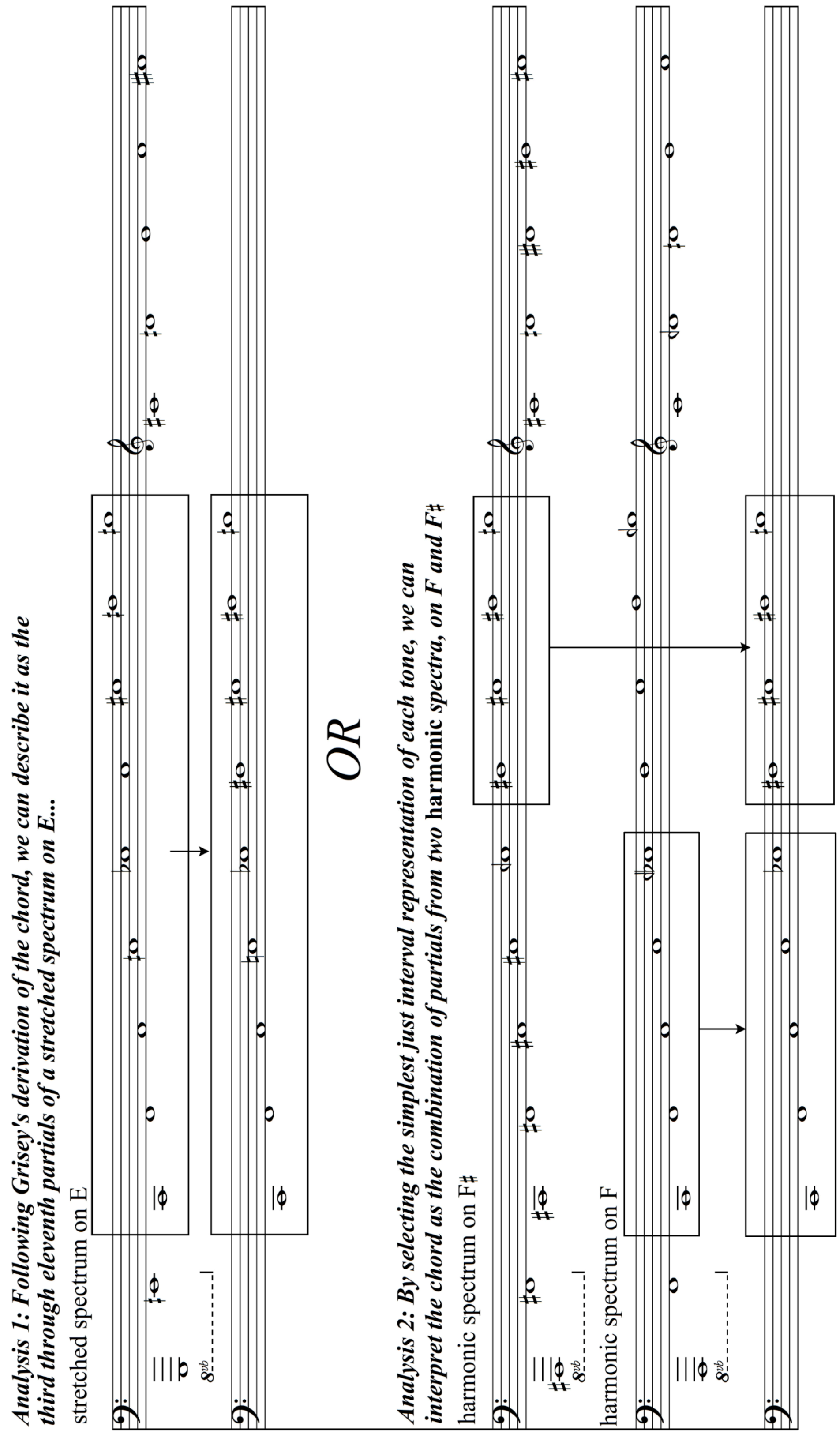

Figure 4.11: analysis of final harmonic area of movement

The complexity and multivalence of these analyses reflect the aural richness of the harmony—I would argue that the competing pull of different tone representations is one of the things that keeps our attention engaged throughout this minute-long, largely static passage. When we view the chord through the lens of tone representation, its derivation from an exaggeratedly stretched spectrum is irrelevant—we focus instead on the complex ways that the harmony plays on our aural intuitions. The choice between the two types of nature I've discussed here—one external, and one internal—illustrates a broader decision between two models of analysis for Grisey's music. One is essentially formalist and philological, based on sketch study and the recreation of the composer's material and ideas, while the other is essentially phenomenological and pragmatic—the analyst's subjective experience of the piece is taken as the essential explicandum. To my mind, the subjectivity of the pragmatic approach is not a weakness, but rather a strength—by allowing such analytical flexibility, we recognize the richness of musical listening and avoid the flattening of experience into a simplistic formal mold.

EPILOGUE: Tone Representation as a Pragmatic Theory

Tone representation is a valuable tool of a pragmatic approach to analysis: an approach based on taking aural experience seriously, instead of retreating to formalistic abstractions or speculations on the composer's intent. By focusing our attention on sonic quality and rootedness, it suggests new readings of music in a variety of styles, from Schoenberg's atonal music to Grisey's spectral works. The theory makes it possible to put into words some of the most elusive aspects of our experience of complex harmonies—and by emphasizing listening instead of abstract "structure," it offers a promising alternative to existing analytical techniques.

Through its emphasis on listening, tone representation can acknowledge ambiguity and multivalence as an essential part of musical experience, rather than insisting on a single structural interpretation. With the preference rules proposed in Chapter 2, we can weigh the advantages of different harmonic interpretations, recognizing that we draw upon multiple tone representations as we listen to a piece. As a theory of listening—as opposed to a theory of musical construction, like serialism or developing variation—tone representation can be applied to a broad range of works, not just music created by a certain compositional method. An analyst applying the tools of tone representation can approach works in different styles with the same interpretive principles: tone representation assumes that we bring the same principles of harmonic listening to all music. This common listening grammar can help us to formulate ideas about contemporary harmony which translate across stylistic boundaries

Tone representation revives (with modifications) the ratio model of interval, which dominated music theory for millennia but has been eclipsed by the competing distance model in twentieth-century theory. In the version of ratio theory proposed here, the Pythagorean/just intonation tradition is updated with concepts from music psychology and psychoacoustics; certain aspects of the theory which once relied on numerology and Platonic forms can now be revised to rest on scientific research. The growing interest by composers in microtonality, spectral harmony, electronic sound, and extended just intonation makes the time ripe for a return to the insights offered by ratio intervals: in particular, their ability to describe specific qualities of consonance and dissonance and root stability. Ratio theory turns the focus away from the analytical geometries of the distance model toward an engagement with the more sensual aspects of musical sound.

The theory of tone representation depends on several basic premises, described in Chapter 1. The first (and most essential) of these premises is that just intervals are the *referential* intervals for harmonic perception: the stable intervals which act as landmarks in the continuum of all possible interval sizes. Embracing this approach to thinking about harmony offers a fresh perspective on pitch structure, with several distinct advantages. One is the potential for limitless expansion of the world of tone relationships: once we accept numerical ratios as basic harmonic building blocks, we can extend the boundaries of our musical systems to include higher prime numbers and more complex just intervals. This potential for expansion offers obvious benefits for composers, but also suggests new ways of understanding our experience of listening to existing repertoires. A second advantage of thinking of pitch in terms of just interval is the organizational framework created by our sense of rootedness. Every interval implies a root, and at the same time an array of related pitches which share that root. This allows us to associate pitches through shared harmonic implications, not just by the geometrical transformations of pitch-class set theory (which takes a purely distance approach to interval). The new harmonic relationships offered by tone representation can broaden our vocabulary for describing musical experience; the flexibility of the method can stand on its own or complement standard distance-based approaches.

One feature that separates tone representation from the tradition of just intonation theory is the acceptance of intonational error. The degree of *tolerance* I've incorporated into the theory is derived from the theories of James Tenney; however, the assumption that we can still recognize just intervals in approximate versions is implicit in the long history of musical temperament. Tolerance for mistuned just intervals allows the application of the theory in a variety of contexts, not just purely-tuned music conceived in just intonation. We find that the harmonic meaning of just intervals persists even in music written for the equal-temperament piano, as we saw in the analysis of an excerpt from Schoenberg's Op. 11, No. 2 at the end of Chapter 2.

The harmonic relationships brought to the fore by tone representation suggest ways of associating pitches in complex musical textures: grouping pitches which are connected by just intervals is a convincing way of segmenting a texture into smaller entities. Frequently I've borrowed terms from Albert Bregman's theory of auditory stream analysis: the parsing of aural information into separate streams of information to form a mental picture of the world. Tone representation's link to auditory scene analysis suggests that it could be a useful tool for an "ecological" approach to discussing music, such as that recently proposed by Eric Clarke.[1] The theory acknowledges that listening to

[1] Eric Clarke, *Ways of listening: an ecological approach to the perception of musical meaning* (Oxford and New York: Oxford University Press, 2005).

music is not simply following a predetermined structure, but a continual, active construction of meaning and perceived form.

My theory of tone representation is implemented in a set of three preference rules, which can be applied to determine the most likely tone representations for any collection of pitches. The rules are designed to work in combination with one another: frequently the demands of one rule will contradict those of another rule, and determining the best tone representation will require a compromise between the two. The preference rules tell us to: 1) prefer tone representations which closely match the music surface, requiring a minimum of mental retuning; 2) prefer the simplest tone representations, i.e., those containing the simplest just intervals; and 3) minimize the number of separate fundamentals. These rules reflect principles from psychological research: the principle of *Prägnanz* in Gestalt psychology, Bregman's rules for auditory scene analysis, and the virtual pitch algorithm of Ernst Terhardt.

As a pragmatic theory, the ultimate test of tone representation is how it can inform our understanding of music: how (as William James puts it) the theory can "help us to get into satisfactory relation with other parts of our experience."[2] Several analyses of twentieth-century music have been offered as illustrations of how the theory might change our understanding of music by composers as diverse as Schoenberg and La Monte Young. The third chapter of this dissertation illustrates the range of composers who have been inspired by the ratio approach to interval; tone representation is particularly apt for exploring music by just intonation composers like Partch, Harrison, Johnston, Sims, Young, Tenney, and others. The theory can also shed light on European composers with more peripheral connections to just intonation theory, like Stockhausen, Ligeti, Stahnke,

[2] William James, "What Pragmatism Means," op. cit., 2

and Haas. It can also provide an alternative take on music by spectral composers like Murail and Grisey; spectral music often draws on concepts (overtones, harmonicity) related to just intonation theory, but through a very different aesthetic and theoretical orientation. In the final chapter of this dissertation, I've used tone representation as a way of understanding Grisey's spectral *Vortex Temporum* through the lens of extended just intonation. Through this approach to the music, we can describe how we make sense of its harmonies without reference to the procedures Grisey used to generate them; the focus of research turns from sketch study and reconstruction of compositional intent to an introspective exploration of musical experience.

This turn toward experience is the goal of the theory of tone representation; as a pragmatic theory grounded in the psychology of musical perception, it offers a clear way to discuss our harmonic intuitions. By emphasizing the act of listening as opposed to an abstract structure, tone representation is an alternative to formalist modes of analysis. As twenty-first century composers continue to explore the physical and sensual properties of sound, tone representation has the potential to become a valuable tool in interpreting and describing our musical experience.

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